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CHAOTIC BEHAVIOUR OF IDEAL FOUR-LEVEL LASER WITH PERIODIC PUMP MODULATION

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In this paper we study the predictability of the ideal four-level laser for different pump modulation frequencies. Time series obtained from the well-known two ODE system are processed using the Grassberger-Procaccia algorithm in order to estimate the correlation dimension and the error-doubling time (computed via the Kolmogorov entropy). Results evidence windows of low predictability, which can be associated with chaotic behavior. The attractor dimension, approximated by the correlation dimension, varies between 1.61 and 2.56 for different pump modulation frequencies. However, these values were obtained in a somehow indefinite or undeterminated way. Supported by the large discrepancies in the results reported when chaotic dynamics is analyzed in other areas of physics (e.g. atmospheric physics), the rather vague character of the criteria used when setting some parameters on which the above-mentioned quantities crucially depend is emphasized.

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1. Introduction

Convincing evidence for deterministic chaos has come from a variety of scientific domains; therefore, the question of detecting and quantifying chaos has become an important one. A very striking aspect is that phenomena which are predictable by their intrinsic nature presents 'windows' of unpredictability, i.e. for some values of a parameter (or more) the system exhibits chaotic behavior. On the other hand, phenomena which are mainly chaotic may have a large predictability in some narrow intervals of these parameters values. The well-known two ODE [1] system describing the ideal four-level laser with periodic pump modulation is numerically integrated using the Crank-Nicholson method with two iterations [2]. Due to computer time limitations, only 1000 data points were retained for each pump modulation frequency ω. Then the Grassberger-Procaccia [3,4] algorithm is used to obtain information about the predictability of the system (measured by the Kolmogorov entropy), and about the correlation dimension, for ω belonging to the interval 0.010-0.100. The discretization interval for ω is 0.002, a rather large value imposed by the time consuming character of the algorithm. However, results evidenced about one order of magnitude alternating values for the error-doubling time (obtained from the Kolmogorov entropy) for some 'windows' of ω. The correlation dimension, which approximates the attractor dimension, ranged between 1.61 and 2.56, the greater then 2 values being explainable by the presence of time in the second equation. But both quantities mentioned above depend crucially on the preset of some parameters appearing in their definitions [3,4]. Unfortunately, the choose of these parameters is subjectively defined [3,5], and therefore we will not emphasize as a conclusion that for some values of ω the attractor has a fractal or an integer dimension, even if the correlation dimension can be 'driven' to be about the expected value of 2. We want to mention here that our discretization for ω is more coarse than in [1]. Nevertheless, the problem of a well-defined method for choosing the data, the time delay and other parameters is an open one, as can be seen from the differences between the results reported in atmospheric sciences [5].

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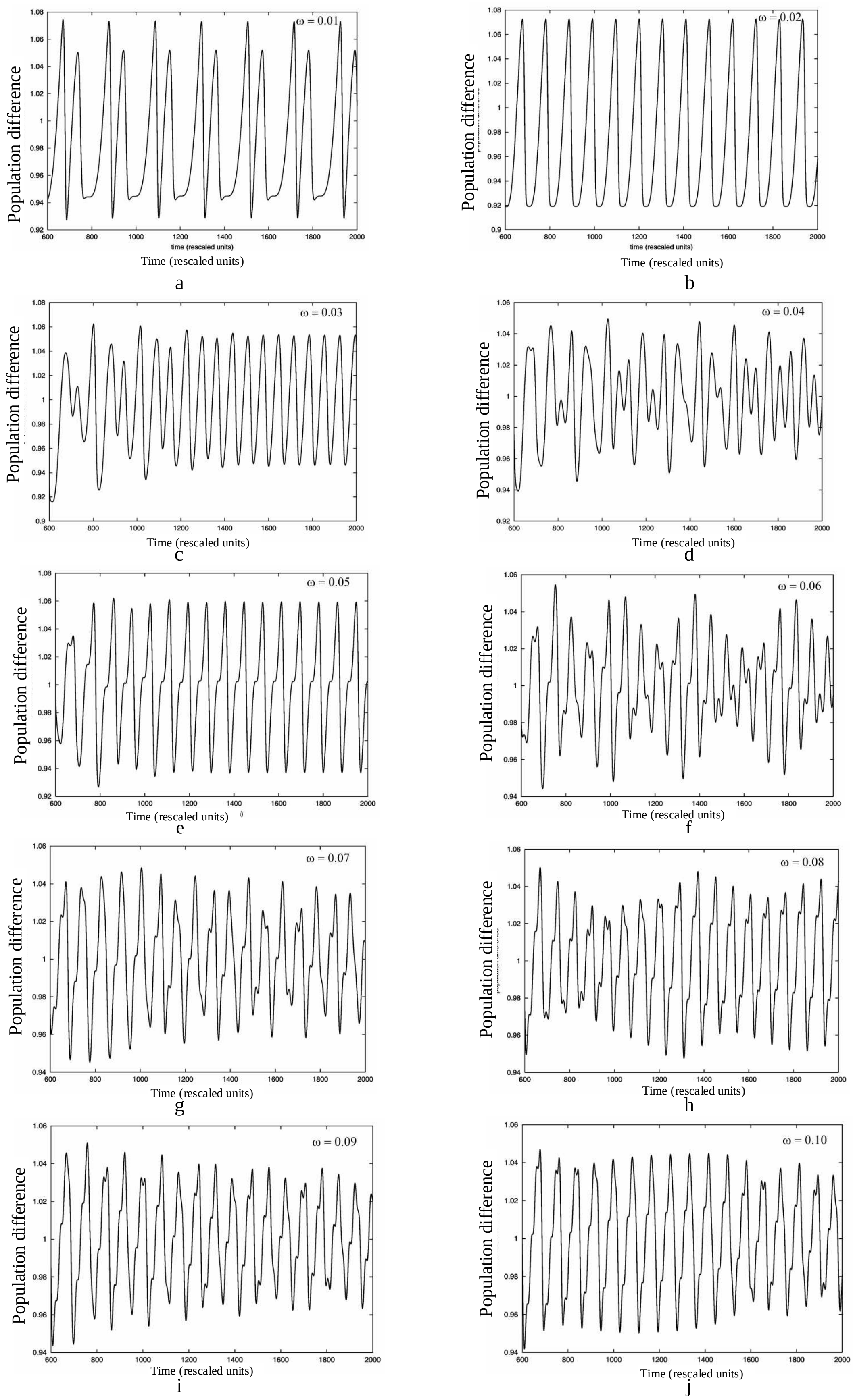

Fig. 1. Population difference vs. time.

2. Numerical experiments

We started with the normalized two ODE describing the four - level laser with periodic pump modulation [1]:

$$\begin{aligned} \dot{q} &= -q + nq + sn \\ \dot{n} &= p_0(1 + p_m \sin \omega t) - nq \end{aligned} \tag{1}$$

Here, p_m is the degree of modulation, $p_m = 1$, p_0 is the constant pump term, $p_0 = 6 \times 10^{-4}$, s is the spontaneous emission rate into the laser mode, $s=10^{-7}$, and ω - the modulation frequency. q and n are the normalized photon number and population difference. These two ODE were numerically integrated using the iterated Crank - Nicholson method with two iterations. The number of iterations was chosen conformal to [2]. Fifty runs were made, corresponding to ω belonging to the interval [0.010 - 0.100], from 0.002 to 0.002 (γ^{-1} rescaled time units, with γ = the decay constant), but only ten of them are displayed in the figures below. The normalized population difference was found to be suggestive for evidencing well - behaved or erratic patterns. Mentioning that distinction between regular and nonregular behavior may be sometimes a very difficult task even for long term integration outputs, we can say that in Figs. 1a,b,c,e, (and h) the signal is periodic (after a transient regime), whereas the others are erratic.

Next the Grassberger - Procaccia algorithm was used to determine the correlation dimension, which approximates the attractor dimension, and the Kolmogorov entropy, which gives the error - doubling time. The reader is referred to [3-5] for details. The transient regime was removed, so only the data between 1000 and 2000 time units from the graphs before were processed. This limited number of data was imposed by the time consuming character of the algorithm, but they appear to be sufficient to evidence large differences in predictability.

3. Results and discussion

We begin this section by stressing that the correlation dimension and the Kolmogorov entropy values depend on the preset of some parameters which enter in their definitions. As for example, the correlation function calculates the density of points on an attractor within a range of distances r from a given point, and then averages this density over all the points. The range r is in turn delimited by some values v_{min}, v_{max}, which have to be chosen as r to be sufficiently large, to overcome the noise, but much lesser than the horizontal extent of the data [3-5]. The values v_{min} and v_{max} satisfying these criteria influences not only the saturation value of the correlation function, but the saturation process itself. Therefore, when we are dealing with real (and then a finite amount of) data, the results must be carefully interpreted. Many runs (maybe not enough) were made for different values of v_{min} and v_{max}, and even when saturation was reached, different results were obtained. Some of them were unrealistic, and are not presented here. Figs. 2 a - j display the attractor dimension vs. the embedding dimension, k.

Each graph was obtained by carefully choosing the values of v_{min} and v_{max}. We see that for ω = 0.010, 0.020, 0.030 and 0.050 the attractor dimension is about the somehow expected value of 2. The greater then 2 (noninteger) values indicate the manifestation of the pump time dependence as a supplementary degree of freedom (see also Fig. 4). However, remembering that the attractor dimension represents the minimum number of variables necessary to describe the time evolution of the system, we will not speculate by associating to this degree of freedom a particular physical phenomenon neglected when deriving eqs. (1) and which could be important in some specific situations.

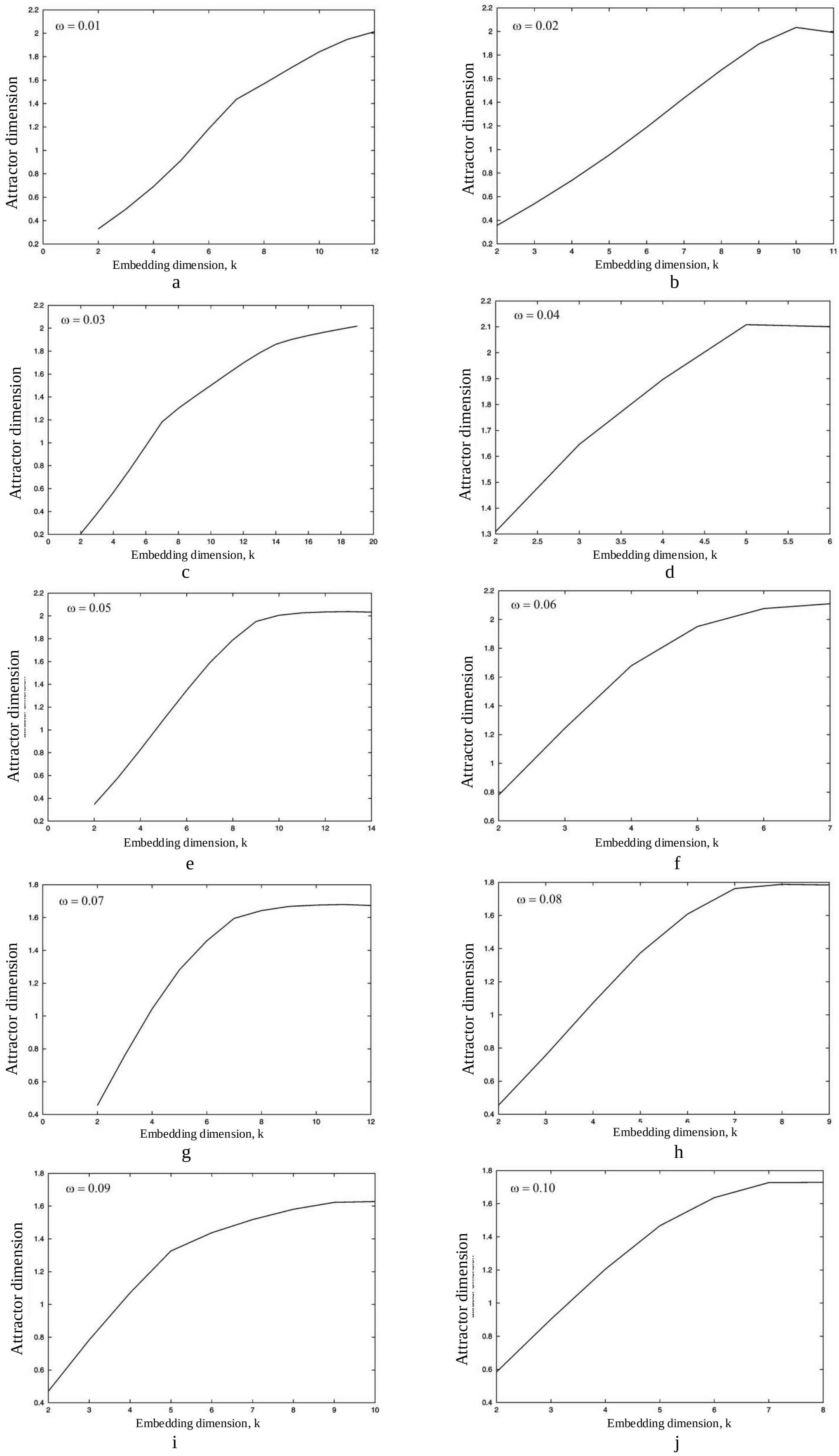

Fig. 2. Attractor dimension versus embedding dimension for various ω.

As for the less than 2 values, they sustain the reliability of the same ODE system mentioned above. Our discussion aims to evidence differences in predictability for different pump frequencies, but it is clear that we cannot fix in an objective way a threshold for the error-doubling time measuring this mathematically calculated quantity. As mentioned above, the same situation holds, even more pronounced, for the signals aspect. These considerations, together with the comments above about the problems appearing in applying the Grassberger-Procaccia algorithm, lead to inadvertencies when we associate, for example, patterns from Figs. 1a-j to error-doubling time values. Therefore, we will discuss only large differences in error-doubling time as relevant. Consequently, from Table 1 below and from Fig. 3, we see that for ω=0.040, 0.060 and 0.090, for which the signal is irregular (Figs. 1d,f,i), the error-doubling times are 3.3, 4.3 and 1.5, respectively. Similarly, to ω=0.030 and 0.050 (Figs. 1c,e) correspond error-doubling times of 17.5 and 20. We shall not try to comment the inadvertencies or the equivoque of other associations.

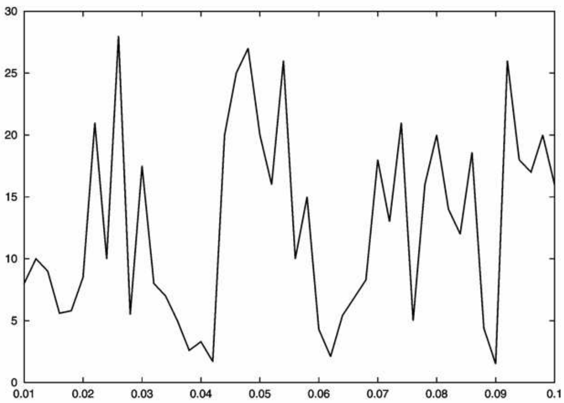

Fig. 3. Error doubling time vs. frequency.

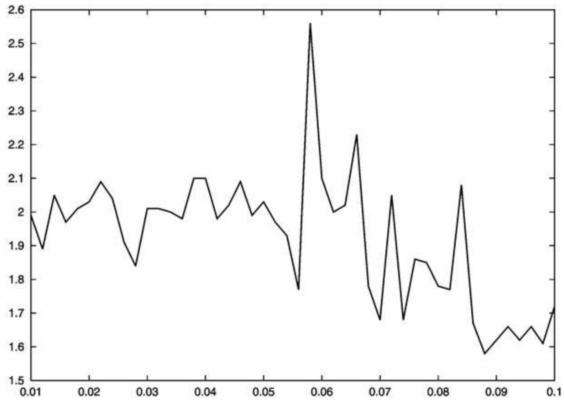

Fig. 4. Attractor dimension vs. frequency.

Table 1 presents some of these values for different ω, together with the minimum of the autocorrelation coefficient, r_{min}, for each data series. By using it together with Figs. 3,4 we observe fractal (both greater and lesser than 2) dimensions for regions centered on 0.04, 0.06 and 0.09 approximately from the ω domain, where the predictability is relatively low. But there are also fractal dimensions which correspond to relatively large predictabilities. This aspect will be discussed in a further article, after using other methods for stability analysis.

Table. 1.

ω (rescaled time units)	r_{min}	attractor dimension	error - doubling time (rescaled time units)
0.010	0.31	1.99	8
0.020	0.34	2.03	8.5
0.030	0.50	2.01	17.5
0.040	0.60	2.10	3.3
0.050	0.45	2.03	20
0.060	0.50	2.10	4.3
0.070	0.60	1.68	18
0.080	0.47	1.78	20
0.090	0.50	1.62	1.5
0.100	0.57	1.72	16

4. Summary

The Grassberger - Proccaccia algorithm was applied to numerical data series obtained from the ODE describing the ideal four - level laser with periodic pump modulation in order to estimate the predictability of this system for different values of modulation frequency. Differences of one order of magnitude in the error - doubling time were observed. We evidenced regions of ω where, as expected, irregular signals and fractal attractor dimensions correspond to low predictability, and regions where periodicity and integer attractor dimensions are characterized by larger predictabilities. The explanation of the inconstancies in some results invoke, after our opinion, too many factors to be outlined here. Ones of them are intrinsic to this paper (as the shortness of the data and the sensitivity of the method used, which were already mentioned), but the whole domain of stability analysis is still in continuous development. Further studies will involve the calculation of the Lyapunov exponents and Kolmogorov entropy defined this time as the sum of all the Lyapunov exponents, for a better predictability estimation. Results will be compared and published elsewhere.

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