The cavity perturbation method for the measurement of the relative dielectric permittivity in the microwave range

S.-B. BALMUS, G.-N. PASCARIU, F. CREANGA, I. DUMITRU, D. D. SANDU
Faculty of Physics, "Al. I. Cuza" University of Iasi, Romania

The cavity (small) perturbation is a very suitable method for the measurement of the dielectric relative permittivity at microwave frequencies. In this paper we give the most important relations of this method, particular relations for rectangular resonant cavities and some experimental results. Also, an analysis of total relative errors and the second order perturbations method are presented.

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1. Introduction

The perturbation of resonant cavities may be achieved by the modification of the volume (shape perturbations) or by the introduction of small pieces of dielectric or ferromagnetic material (material perturbations). In the case of dielectric measurements, the second way is applied. Consider that a original cavity (Fig. 1a) is perturbed by a change in the permittivity or permeability (Fig. 1b).

The change of the resonant frequency of the perturbed cavity due to material perturbation is given by the relation [1], [2], [3]

$$\frac{\omega_2-\omega_1}{\omega_2}=\frac{-\int_{V_C}\left[(\varepsilon_2-\varepsilon_1)\vec{E}_1^*\cdot\vec{E}_2+(\mu_2-\mu_1)\vec{H}_1^*\cdot\vec{H}_2\right]dV}{\int_{V_C}\left(\varepsilon_1\vec{E}_1^*\cdot\vec{E}_2+\mu_1\vec{H}_1^*\cdot\vec{H}_2\right)dV} \tag{1}$$

where: $\vec{E}_1, \vec{H}_1, \omega_1, \mu_1$ and ε_1 characterize the unperturbed cavity; $\vec{E}_2, \vec{H}_2, \omega_2$, $\mu_2 = \mu_1 + \Delta\mu$ and $\varepsilon_2 = \varepsilon_1 + \Delta\varepsilon$ characterize the perturbed cavity.

This relation is an exact equation for the change in resonant frequency but it is not a very usable form since we generally do not know the exact fields $\vec{E}_2$, $\vec{H}_2$ in the perturbed cavity. In the case of measurements a lot of approximations were proposed. The most spread approximation considers very small material samples compared with the cavity volume; in these cases we have *small perturbations* of the cavity.

In this work we are dealing with very small dielectric samples. Therefore we can consider that the measurement method is based on *small perturbations of the resonant cavity*.

Cavity perturbation measurements can be highly accurate and are particularly advantageous in the determination of relative permittivity of dielectrics with small loss tangents. Perturbation techniques permit the measurement of dielectric samples of small sizes and various shapes. The most convenient of the shapes are the spheres, rods, discs and slabs.

2. Approximations for thesmall perturbations of the resonant cavity

In our paper we consider that the first cavity 1 is empty ($\mu_1 = \mu_0$ and $\varepsilon_1 = \varepsilon_0$) (Fig. 1a) and the volume V_S of the sample is very small compared with the volume V_C of the cavity (Fig. 1b), therefore we can assume that $\vec{E}_2 = \vec{E}_1$ and $\vec{H}_2 = \vec{H}_1$ in the integrands of denominator of eq. (1).Under these conditions this equation can be rewritten as follows

$$\frac{\omega_2-\omega_1}{\omega_2}=\frac{-\int_{V_S}(\varepsilon_2-\varepsilon_0)\vec{E}_1^*\cdot\vec{E}_2\,dV-\int_{V_S}(\mu_2-\mu_0)\vec{H}_1^*\cdot\vec{H}_2\,dV}{2\varepsilon_0\int_{V_C}\left|\vec{E}_1\right|^2dV} \tag{2}$$

where the integrands of the denominator become $\left|\vec{E}_1\right|^2$ and $\left|\vec{H}_1\right|^2$ respectively, and the resonance equality

$$\varepsilon_0\int_{V_C}\left|\vec{E}_1\right|^2dV=\mu_0\int_{V_C}\left|\vec{H}_1\right|^2dV \tag{3}$$

was used. The integrations in the nominator of eq. (2) are performed only over volume V_S of the sample since in the cavity 2 we have $\mu_2 = \mu_0$ and $\varepsilon_2 = \varepsilon_0$ except the small volume V_S.

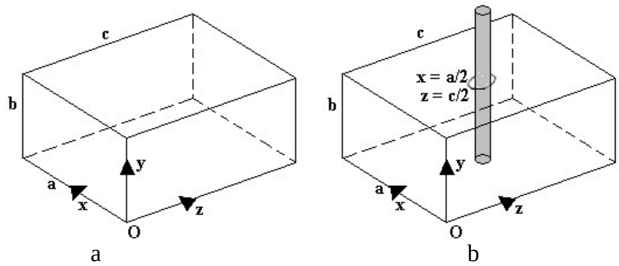

Fig. 1. a- empty cavity; b- introduction of the dielectric sample.

When the sample is assumed non-magnetic, $\mu_2 = \mu_0$, the second term of the nominator in eq. (2) drops out and we obtain

$$\frac{\omega_2 - \omega_1}{\omega_2} = -\frac{\varepsilon_{r2} - 1}{2} \frac{\int_{V_S} \vec{E}_1^* \cdot \vec{E}_2 \, dV}{\int_{V_C} \left|\vec{E}_1\right|^2 dV} \tag{4}$$

where: $\varepsilon_{r2} = \varepsilon_2 / \varepsilon_0$ is the relative (complex) dielectric permittivity of the sample; $\vec{E}_1$ is the field in the empty (non-perturbed) cavity; $\vec{E}_2$ is the field in the sample and is determined by the shape and the size of samples.

Generally the empty cavity and dielectric materials have losses. Therefore the angular frequency ω associated with a dissipative system is a complex quantity and can be written as [4]

$$\omega = \omega_R + j\omega_J \tag{5}$$

and the overall quality factor Q_T is defined as

$$Q_T = \frac{\omega_R}{2\omega_J} \tag{6}$$

Consider the expression

$$\frac{\delta\omega}{\omega} = \frac{\omega_2 - \omega_1}{\omega_2} \tag{7}$$

where: both ω_1 and ω_2 are complex in the sense of eq. (5); $\omega_{R1} \cong \omega_{R2}$ and $\omega_J << \omega_R$. On expanding $\delta\omega/\omega$ and taking into account these approximations, we have

$$\frac{\delta\omega}{\omega} = \frac{(\omega_{R2} - \omega_{R1}) + j(\omega_{J2} - \omega_{J1})}{\omega_{R2}\left(1 + j\frac{\omega_{J2}}{\omega_{R2}}\right)} = \left[\left(\frac{f_{R2} - f_{R1}}{f_{R2}}\right) + j\left(\frac{1}{2Q_{T2}} - \frac{1}{2Q_{T1}}\right)\right]\left[1 - j\frac{1}{2Q_{T2}}\right] \tag{8}$$

where $f_{R2} = \omega_{R2}/2\pi$, $f_{R1} = \omega_{R1}/2\pi$.

Since $1/2Q_{T2}$ can be neglected compare with unity we can write

$$\frac{\delta\omega}{\omega} = \frac{f_{R2} - f_{R1}}{f_{R2}} + \frac{j}{2}\left(\frac{1}{Q_{T2}} - \frac{1}{Q_{T1}}\right) \tag{9}$$

This equation provides the link between the measured quantities, f_R and Q_T, and the theoretical expressions involving $\delta\omega/\omega$ which are considered in eq. (4). By substitution of eq. (9) in eq. (4) we obtain the small perturbation formula

$$2\frac{f_{R2} - f_{R1}}{f_{R2}} + j\left(\frac{1}{Q_{T2}} - \frac{1}{Q_{T1}}\right) = -(\varepsilon_{r2} - 1)\frac{\int_{V_S} \vec{E}_1^* \cdot \vec{E}_2 \, dV}{\int_{V_C} \left|\vec{E}_1\right|^2 dV} \tag{10}$$

If we consider a filling coefficient defined by the relation [3]

$$N = \frac{\int_{V_S} \vec{E}_1^* \cdot \vec{E}_2 \, dV}{\int_{V_C} \left|\vec{E}_1\right|^2 dV} \tag{11}$$

eq. (10) can be rewritten as follows

$$2\frac{f_{R2} - f_{R1}}{f_{R2}} + j\left(\frac{1}{Q_{T2}} - \frac{1}{Q_{T1}}\right) = -(\varepsilon_{r2} - 1)N \tag{12}$$

In almost all practical cases ε_{r2} is a complex quantity $\varepsilon_{r2} = \varepsilon'_{r2} - j\varepsilon''_{r2}$, therefore we have

$$\varepsilon'_{r2} = 1 + \frac{2}{N}\frac{f_{R1} - f_{R2}}{f_{R2}}, \quad \varepsilon''_{r2} = \frac{1}{N}\left(\frac{1}{Q_{T2}} - \frac{1}{Q_{T1}}\right) \tag{13}$$

The filling coefficient may be determined if the fields in the cavities are known before and after the perturbation; also it may be determined experimentally using a sample with known ε'_{r2} and ε''_{r2}.

3. Particular results for the rectangular cavities

3.1. Experimental setup

The experimental setup is presented in Fig. 2.

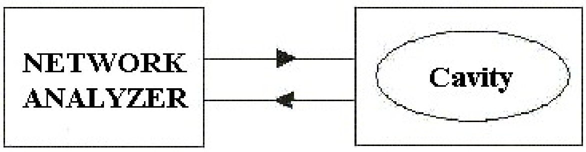

Fig .2. Experimental installation (Hewlett packard 8714C 300KHz-3000MHz RF network analyzer).

In our experiment we used the *small perturbations method* for a rectangular cavity, having the dimensions *a = 5.8 cm, b = 2.5 cm* and *c =9.16 cm* and which is oscillating on the TE_{101} mode at *3GHz*. The cavity, which initially was empty $\varepsilon_1 = \varepsilon_0$, is perturbed by the introduction at *x=a/2, z=b/2* and *y=0÷b* of a cylindrical dielectric sample having the radius *r* and relative dielectric permittivity ε_{r2}. After the perturbation the oscillation mode and the field lines are unchanged but the resonance frequency f_r and the quality factor *Q* decrease.

3.2. The permittivity expressions

The electrical field for the TE_{101} mode has only one component [4]

$$E_y = -jE_{0y}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{c}z\right) \quad (14)$$

where E_{0y} is a real constant depending on the applied signal

Assuming that the electrical field inside the dielectric sample ($E_2 = E_1$) is constant

$$E_1 = -jE_{0y}\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = -jE_{0y} \quad (15)$$

the filling coefficient (11) becomes

$$N = \frac{4\pi r^2}{ac} \quad (16)$$

Therefore, for the complex permittivity we have

$$\varepsilon'_{r2} = 1 + \frac{ac}{2\pi r^2}\frac{f_{1r} - f_{2r}}{f_{2r}} \quad (17)$$

$$\varepsilon''_{r2} = \frac{ac}{4\pi r^2}\left(\frac{1}{Q_2} - \frac{1}{Q_1}\right) \quad (18)$$

3.3. Experimental results

Some experimental results obtained from the measurements of some usual materials used in the microwave range and some epoxy resins are presented in Table 1 and Table 2.

Table 1. Results obtained for some usual materials (resonance frequency f_r=3GHz).

Sample	Empty cavity	Teflon (PTFE)	Methacrylate (PMMA)	Erthalon 12	Eralyte TX	PVC Ivory
Radius (mm)		2.5	2.5	2.5	2.5	2.5
N	0	0.0148	0.0148	0.0148	0.0148	0.0148
ε'_r	1	2.05	2.50	2.94	2.78	2.61
ε''_r	0	0.0003	0.0005	0.060	0.0006	0.020

Table 2. Result obtained for some epoxy resins (resonance frequency f_r=3GHz).

Sample	Epoxy resin + Polyurethane	Epoxy resin + Triethylene-tetrachloride-amine	Epoxy resin + Bismaleimide $C_{21}H_{14}N_2O_4$	Epoxy resin + Diaminodiphenyl-methane $C_{13}H_{14}N_2$
Radius (mm)	1.45	1.45	1.45	1.45
N	0.005	0.005	0.005	0.005
ε'_r	3.12	3.24	3.10	3.26
ε''_r	0.205	0.21	0.14	0.208

The values of the permittivity obtained in our experiments are in good agreement with the values given in the literature (handbooks and papers). In Table 3 are presented the catalog values [6] of the permittivity of some materials that we used in our experiments.

Table 3. The catalog values for some material used in our experiments.

Sample	Teflon (PTFE)	Erthalon 12	Methacrylate (PMMA)
ε'_r in reference [6]	2.0÷2.1	2.9	2.6
ε'_r in our experiments	2.05	2.94	2.5

3.4. Relations for the calculus of total relative errors

Applying the logarithm to the relation (17) and separating the terms we obtain

$$ln(\varepsilon'_{r2} - 1) = ln\frac{1}{2\pi} - 2\ln r + \ln a + \ln c + ln\left(\frac{f_{1r} - f_{2r}}{f_{2r}}\right) \quad (19)$$

Calculating the differential of this relation we have

$$\frac{d\varepsilon'_{r2}}{\varepsilon'_{r2} - 1} = -2\frac{dr}{r} + \frac{da}{a} + \frac{dc}{c} + \frac{1}{f_{1r} - f_{2r}}df_{1r} + \frac{f_{1r}}{f_{2r}(f_{2r} - f_{1r})}df_{2r} \quad (20)$$

Using that $f_{1r} > f_{2r}$, we pass to finite differences

$$\frac{\Delta\varepsilon'_{r2}}{\varepsilon'_{r2} - 1} = 2\frac{\Delta r}{r} + \frac{\Delta a}{a} + \frac{\Delta c}{c} + \frac{1}{f_{1r} - f_{2r}}\Delta f_{1r} + \frac{f_{1r}}{f_{2r}(f_{1r} - f_{2r})}\Delta f_{2r} \quad (21)$$

Because

$\Delta f_{1r} = \Delta f_{2r} = \Delta f$, the measurement errors are the same for each frequency (22)

the expression (21) becomes

$$\frac{\Delta\varepsilon'_{r2}}{\varepsilon'_{r2} - 1} = 2\frac{\Delta r}{r} + \frac{\Delta a}{a} + \frac{\Delta c}{c} + \frac{f_{1r} + f_{2r}}{f_{2r}(f_{1r} - f_{2r})}\Delta f \quad (23)$$

The measurement errors for the cavity dimensions are $\Delta a = \Delta c = 10^{-4}m = 10^{-2}cm$ (*a = 5.8 cm,* and *c = 9.16 cm)* which means that the terms containing the dimensions of the cavity give errors which are smaller then *0.002;* therefore they can be ignored

$$\frac{\Delta\varepsilon'_{r2}}{\varepsilon'_{r2} - 1} \cong 2\frac{\Delta r}{r} + \frac{f_{1r} + f_{2r}}{f_{2r}(f_{1r} - f_{2r})}\Delta f \quad (24)$$

Using a HEWLETT PACKARD 8714C 300 KHz – 3000 MHz RF NETWORK ANALYZER the measurement error for frequencies is $\Delta f = 10^{-3}\, MHz$, $f_{1r}\, f_{2r} \cong 3000\, MHz$ and $f_{1r} - f_{2r} \cong 20 MHz$. This means that the approximate value of the second right term in eq (24) is 10^{-3} - 10^{-4} and we can ignore this term compared with the first one.

$$\frac{\Delta\varepsilon'_{r2}}{\varepsilon'_{r2} - 1} \cong 2\frac{\Delta r}{r} \tag{25}$$

In order to calculate the relative errors for the imaginary part of permittivity we apply the logarithm to the relation (18), and separating the terms we have

$$\ln \varepsilon''_{r2} = \ln\frac{1}{4\pi} - 2\ln r + \ln a + \ln c + \ln\left(\frac{1}{Q_2} - \frac{1}{Q_1}\right) \tag{26}$$

If we calculate the differential of the last relation, we obtain

$$\frac{d\varepsilon''_{r2}}{\varepsilon''_{r2}} = -2\frac{dr}{r} + \frac{da}{a} + \frac{dc}{c} + \frac{Q_2}{Q_1 - Q_2}\frac{dQ_1}{Q_1} + \frac{Q_1}{Q_2 - Q1}\frac{dQ_2}{Q_2} \tag{27}$$

Using that $Q_1 > Q_2$ and passing to finite differences we have

$$\frac{\Delta\varepsilon''_{r2}}{\varepsilon''_{r2}} = 2\frac{\Delta r}{r} + \frac{\Delta a}{a} + \frac{\Delta c}{c} + \frac{Q_2}{Q_1 - Q_2}\frac{\Delta Q_1}{Q_1} + \frac{Q_1}{Q_1 - Q_2}\frac{\Delta Q_2}{Q_2} \tag{28}$$

The quality factor is given by

$$Q = \frac{f_r}{f_d - f_s} \tag{29}$$

where: f_r is the resonance frequency; f_d is the right frequency and f_s is the left frequency at $-3dB$ attenuation of the resonance curve.

We apply the logarithm and separate the terms

$$\ln Q = \ln f_r - \ln(f_d - f_s) \tag{30}$$

By calculating the differential of the last expression we have

$$\frac{dQ}{Q} = \frac{df_r}{f_r} - \frac{1}{f_d - f_s}df_d + \frac{1}{f_d - f_s}df_s \tag{31}$$

Using the inequality $f_d > f_s$ and considering that the measurement errors are the same for all frequencies $\Delta f_r = \Delta f_d = \Delta f_s = \Delta f$, we pass to finite differences

$$\frac{\Delta Q}{Q} = \left(\frac{1}{f_r} + \frac{2}{f_d - f_s}\right)\Delta f \tag{32}$$

Thus, relation (28) becomes

$$\frac{\Delta\varepsilon''_{r2}}{\varepsilon''_{r2}} = 2\frac{\Delta r}{r} + \frac{\Delta a}{a} + \frac{\Delta c}{c} + \frac{Q_2}{Q_1 - Q_2}\left(\frac{1}{f_{1r}} + \frac{2}{f_{1d} - f_{1s}}\right)\Delta f + \frac{Q_1}{Q_1 - Q_2}\left(\frac{1}{f_{2r}} + \frac{2}{f_{2d} - f_{2s}}\right)\Delta f \tag{33}$$

The measurement errors for the cavity dimensions are $\Delta a = \Delta c = 10^{-4} m = 10^{-2} cm$ (*a = 5.8 cm,* and *c =9.16 cm)*; this means that the terms containing the dimensions of the cavity give errors which are smaller then *0.002* and so they can be ignored

$$\frac{\Delta\varepsilon''_{r2}}{\varepsilon''_{r2}} \cong 2\frac{\Delta r}{r} + \frac{Q_2}{Q_1 - Q_2}\left(\frac{1}{f_{1r}} + \frac{2}{f_{1d} - f_{1s}}\right)\Delta f + \frac{Q_1}{Q_1 - Q_2}\left(\frac{1}{f_{2r}} + \frac{2}{f_{2d} - f_{2s}}\right)\Delta f \tag{34}$$

As specified above, the measurement error for frequencies is $\Delta f = 10^{-3}\, MHz$, $f_{1r}\, f_{2r} \cong 3000\, MHz$ and $f_d - f_s \cong 10 MHz$. This means that the approximate value of the second and third right terms in eq. (34) is 10^{-3} - 10^{-4} and we can ignore these terms compare with the first one

$$\frac{\Delta\varepsilon''_{r2}}{\varepsilon''_{r2}} \cong 2\frac{\Delta r}{r} \tag{35}$$

In our experiments the measurement error of the sample radius is $\Delta r = 0.05\, mm$ and the radius of the samples is $r = 1.45 \div 2{,}5\ mm$. This means that the maximum value of the total measurement error for the complex permittivity is about 4÷7 %.

$$\left.\begin{aligned}\frac{\Delta\varepsilon'_{r2}}{\varepsilon'_{r2} - 1} &\cong 2\frac{\Delta r}{r}\\ \frac{\Delta\varepsilon''_{r2}}{\varepsilon''_{r2}} &\cong 2\frac{\Delta r}{r}\end{aligned}\right\} = 0.04 \div 0.07 \tag{36}$$

The maximum total measurement error decreases when the radius of the sample increases. Increasing the radius means that also the filling coefficient N increases. The superior limit for *N* is 0.1 because above this value the approximations made for the small perturbations method are no more correct.

4. The second order perturbation method

The conventional small perturbations method presented in the previous sections is applicable only when the sample determines a small perturbation of the cavity field. Sometimes the material is very fragile and the technological obtaining of the samples with small dimensions is very difficult. In this section we will present a method for the determination of the dielectric permittivity with smaller errors by taking into account the superior modes in a second order perturbation formula [7].

We consider an ideal resonant cavity of a certain shape which is limited by the domain D. Using the time dependence $exp(-j\omega t)$ the unperturbed equation which characterize the intensity of the electric field for each mode is

$$\left(\nabla^2 - k_\alpha^2\right)\vec{E}_\alpha = 0 \qquad (37)$$

where α is the mode number obtained by arranging the modes from small frequencies to high frequencies; k_α is the resonance wave number of the α mode.

We suppose that the resonance modes TE and TM in the cavity are orthonormate and orthogonal

$$\begin{gathered}\left\langle \vec{E}_m^{TE}, \vec{E}_n^{TE} \right\rangle = \iiint_V \vec{E}_m^{TE} \cdot \vec{E}_n^{TE}\, dV = \delta_{mn} \\ \left\langle \vec{E}_m^{TM}, \vec{E}_n^{TM} \right\rangle = \delta_{mn} \\ \left\langle \vec{E}_m^{TE}, \vec{E}_n^{TM} \right\rangle = P_{mn}\delta_{mn}\end{gathered} \qquad (38)$$

Generally we consider that $\left\langle \vec{E}_m, \vec{E}_n \right\rangle = P_{mn}\delta_{mn}$; $P_{mn} = 1$ when the type of the resonance modes is the same.

If the cavity is perturbed by the introduction of a dielectric sample limited by the domain D_1 and having the dielectric permittivity ε_r the resonance equation becomes

$$\left\{\nabla^2 - k^2\left[1 + \lambda\varepsilon_r \chi(\vec{x})\right]\right\}\vec{E} = 0 \qquad (39)$$

where *k* is the resonance wave number of the perturbed cavity, λ is a perturbation coefficient used for the visualization of the perturbation order and

$$\chi(\vec{x}) = \begin{cases} 1, & \vec{x} \in D_1 \\ 0, & \vec{x} \in D - D_1 \end{cases} \qquad (40)$$

The electric field of the perturbed cavity can be developed in function of the orthogonal intensities of the electric field for the unperturbed cavity

$$\vec{E} = \sum_{\alpha=1}^{\infty} c_\alpha \vec{E}_\alpha \qquad (41)$$

Using booth resonance equations we can write

$$\sum_{\alpha=1}^{\infty} c_a \left\{\left(k^2 - k_\alpha^2\right)\vec{E}_\alpha + k^2\lambda\varepsilon_r \chi(\vec{x})\vec{E}_\alpha\right\} = 0 \qquad (42)$$

Multiplying the last relation by $\vec{E}_\beta$ and integrating over the volume of the cavity we obtained the equation for the determination of field amplitude

$$\sum_{\alpha=1}^{\infty} c_a \left\{\left(k^2 - k_\alpha^2\right)\left\langle \vec{E}_\alpha, \vec{E}_\beta \right\rangle + k^2\lambda\varepsilon_r \chi(\vec{x})\left\langle \vec{E}_\alpha, \vec{E}_\beta \right\rangle\right\} = 0 \qquad (43)$$

which for a fixed β gives

$$c_\beta\left(k^2 - k_\beta^2\right)P_{\beta\beta} + k^2\lambda\varepsilon_r \sum_{\alpha=1}^{\infty} c_\alpha U_{\alpha\beta} = 0 \qquad (44)$$

where $U_{\alpha\beta} = \left\langle \vec{E}_\alpha \middle| \chi(\vec{x}) \middle| \vec{E}_\beta \right\rangle$ and c_α are the unknown coefficients.

To solve the last system of equations we must cancel out the determinant

$$\left|\left(k^2 - k_\beta^2\right)P_{\alpha\beta}\delta_{\alpha\beta} + k^2\lambda\varepsilon_r U_{\alpha\beta}\right| = 0 \qquad (45)$$

The development of the last determinant is difficult and in order to solve equation (43) we use an iterative method. If the perturbation is removed we suppose that the solution reduce to $\vec{E}_n$ and $c_n = 1$. Separating the term corresponding to the unknown coefficient c_p we obtain

$$\begin{aligned}&\left[\left(k^2 - k_p^2\right)P_{pp} + \lambda k^2\varepsilon_r U_{pp}\right]c_p = \\ &= -\lambda k^2\varepsilon_r U_{pn} - \lambda k^2\varepsilon_r \sum_{q\neq np} c_p U_{pq}\end{aligned} \qquad (46)$$

In order to develop a method of successive approximations we must write the equation for the determination of c_q. To avoid the repetition we separate the terms in *p* and *n*

$$\begin{aligned}&\left[\left(k^2 - k_q^2\right)P_{qq} + \lambda k^2\varepsilon_r U_{qq}\right]c_{qq} = \\ &= -\lambda k^2\varepsilon_r U_{qn} - \lambda k^2\varepsilon_r c_p U_{qp} - \lambda k^2\varepsilon_p \sum_{r\neq npq} c_r U_{qr}\end{aligned} \qquad (47)$$

Till now no approximations were done. The solution of last equations is obtained by eliminating the sum from the N^th^ equation, solving this equation for q_N and introduction of the result in the (N-1)^th^ equation. For $N = 3$ we eliminate the sum from (46) and we find

$$\begin{aligned}c_q = &-\lambda k^2\varepsilon_r \frac{U_{qn}}{\left(k^2 - k_q^2\right)P_{qq} + \lambda k^2\varepsilon_r U_{qq}} - \\ &-\lambda k^2\varepsilon_r \frac{c_p U_{qp}}{\left(k^2 - k_q^2\right)P_{qq} + \lambda k^2\varepsilon_r U_{qq}}\end{aligned} \qquad (48)$$

By substituting in (46) we obtain

$$\begin{aligned}&\left[\left(k^2 - k_p^2\right)P_{pp} + \lambda k^2\varepsilon_r U_{pp} - \lambda^2 k^4\varepsilon_r^2 \sum_{q\neq np} \frac{U_{qp}U_{pq}}{\left(k^2 - k_q^2\right)P_{qq} + \lambda k^2\varepsilon_r U_{qq}}\right]c_p = \\ &= -\lambda k^2\varepsilon_r U_{pn} + \lambda^2 k^4\varepsilon_r^2 \sum_{q\neq np} \frac{U_{qn}U_{pq}}{\left(k^2 - k_q^2\right)P_{qq} + \lambda k^2\varepsilon_r U_{qq}}\end{aligned} \qquad (49)$$

The equation which gives as k^2 is obtained by making $q = n$ and substituting $c_n = 1$

$$\left(k^2 - k_n^2\right)k^2 = -\lambda\varepsilon_r k^2 U_{nn} + \lambda\varepsilon_r k^2 \sum_{p\neq n} c_p U_{np} \quad (50)$$

Using c_p from (42) we get

$$k^2 = k_n^2 - \lambda\frac{\varepsilon_r k^2 U_{nn}}{P_{nn}} + \quad (51)$$

$$+\lambda^2 \frac{\varepsilon_r^2 k^4}{P_{nn}} \sum_{p\neq n} \frac{U_{np}U_{pn}}{\left[\left(k^2 - k_p^2\right)P_{pp} + \lambda k^2 \varepsilon_r U_{pp} - \lambda^2 k^4 \varepsilon_r^2 \sum_{q\neq np} \frac{U_{qp}U_{pq}}{\left(k^2 - k_q^2\right)P_{qq} + \lambda k^2 \varepsilon_r U_{qq}}\right]}$$

From the last development we keep only the terms till the second order [8]

$$k^2 = k_n^2 - \lambda\frac{\varepsilon_r k^2 U_{nn}}{P_{nn}} + \lambda^2 \frac{\varepsilon_r^2 k^4}{P_{pp}} \sum_{p\neq n} \frac{U_{np}U_{pn}}{\left(k^2 - k_p^2\right)P_{pp}} \quad (52)$$

By making $\lambda = 1$ we obtain the expression of the dielectric permittivity corresponding to the resonance frequency for which we make the determination by solving the second order equation

$$-\varepsilon^2 \sum_{p\neq n} \frac{U_{np}U_{pn}}{\left(1 - \frac{f_p^2}{f^2}\right)P_{pp}} + \varepsilon U_{nn} + \left(1 - \frac{f_p^2}{f^2}\right)P_{nn} = 0 \quad (53)$$

We consider the particular case of a rectangular resonant cavity having the dimensions *a*, *b* and *c*. We normalize the electric fields in the unperturbed cavity

a. TM

$$E_z = \frac{1}{K_M} sin(k_x x) sin(k_y y) sin(k_z z)$$

$$E_x = -\frac{k_x k_z}{K_M \cdot k_l} cos(k_x x) sin(k_y y) sin(k_z z) \quad (54)$$

$$E_y = -\frac{k_y k_z}{K_M \cdot k_l} sin(k_x x) cos(k_y y) sin(k_z z)$$

b. TE

$$E_x = \frac{k_y}{K_E} cos(k_x x) sin(k_y y) sin(k_z z)$$

$$E_y = \frac{k_y}{K_E} sin(k_x x) cos(k_y y) sin(k_z z) \quad (55)$$

where

$$k_x = m\pi / a, k_y = n\pi / a \quad and \quad k_z = p\pi / c.$$

$$k_l = \sqrt{k_x^2 + k_y^2}, \; k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$$K_E = \sqrt{\frac{abc}{8} \cdot \left[k_x^2\left(1 + \delta_{n0}\right) + k_y^2\left(1 + \delta_{m0}\right)\right]} \quad (56)$$

$$K_M = \sqrt{\frac{abc}{8} \cdot \left[\frac{k_x^2 k_z^2}{k_l^4}\left(1 + \delta_{n0}\right) + \frac{k_y^2 k_z^2}{k_l^4}\left(1 + \delta_{m0}\right) + \left(1 + \delta_{p0}\right)\right]} \quad (57)$$

The method was tested by comparison with the classical first order perturbations method for a rectangular cavity having the dimensions *21.2 x 12.1 x 203 mm* and which is oscillating on the fundamental mode. The sample was a dielectric parallelepiped introduced there where the electric field is maxim. We observed the frequency deviations for the fundamental mode and for other five superior modes. The comparative results obtained at the INFIM Institute, Bucharest, Romania, are presented in Table 4.

Table 4. Comparative results for the second order perturbation method

Sample dimensions (mm)	1st order perturbs. ε_r'	2nd order perturbs. ε_r'
1.1 × 1.1 × 12.1	36.3	36.5
1.4 × 1.4 × 12.1	36.9	36.3
1.9 × 1.9 × 12.1	38.3	36.2

5. Conclusions

The most important relations used in the classical “small perturbations method” and in the new “second order perturbation method” are presented.

The values of the permittivity obtained in our experiments are in good agreement with the values from the literature (catalogs and papers).

The maximum range of errors obtained from the calculus relations is 4÷7 %. These small values are good arguments that the cavity small perturbations method is a very suitable method for the measurement of the dielectric relative permittivity at microwave frequencies for dielectrics with small losses.

The most important factor that influences the error is the sample radius r. The others factors: frequencies, dimensions of the cavity and quality factor determine errors which could be ignored compared with the errors given by measurement errors of the sample radius.

For samples with high permittivities or large size the second order perturbation method gives better results.

References

[1] D. M. Pozar, “Microwave Engineering”, Addison-Wesley Publ. Comp. Massachusetts, second ed. 1993.

[2] M. Sucher, J. Fox “Microwave Measurements” third edition, vol. III, Polytechnic Press, Brooklyn, a division of J. Wiley and Sons, N.Y., London, 1963, Chapter IX Dielectric Constant (H. M. Altschuler).

[3] D. D. Sandu, S-B. Balmus, O-G. Aavadanei, G-N. Pascariu “Measurement of the relative dielectric permittivity in the microwave range by the cavity perturbation method” (Communication at the 8th

TENSOR Society Conference, August 2005, Varna, Bulgaria) (in press in Tensor Review).
[4] D. D. Sandu "Microwaves. Physical principles" (in romanian), VICTOR Publishing House, Bucharest, 2005.
[5] Angot "Compléments de mathématiques à l'usage des ingénieurs de l'électrotechnique et des télécommunications", 1961.
[6] J. Brandrup, E.H. Immergut "Polymer Handbook, third edition, Wiley-Interscience Publ., John Wiley & Sons, New York, 1989.
[7] A. El Rafhi "Etude de l'évolution des propriétés diélectriques et magnétiques des matériaux sous champ micro-onde", thèse INPT, 1997.
[8] I. Dumitru, C. Goiceanu "The resonant cavity perturbation method applied for the superior oscillation modes", Romanian Academy, Iasi branch, communication, 2002.

* Corresponding author: sbalmus@stoner.phys.uaic.ro