JOURNAL OF OPTOELECTRONICS AND ADVANCED MATERIALS Vol. 8, No. 4, August 2006, p. 1381 - 1390

The inverse problem in scattering theory of optical fields

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A general model was built for spatial solitons in photorefractive crystals using the inverse problem in the scattering theory. The inverse problem in the scattering theory is defined knowing the spectral data that characterize the scattering. We present a formalism regarding the use of the inverse method in solving the nonlinear differential equations. Envelope singular analytical solutions (solitons) and asymptotically solutions of the wave equation for integral equation (of SBS type) were obtained. The results are in good agreement with the results obtained in other papers.

(Received May 9, 2006; accepted July 20, 2006)

Keywords: Inverse problem, Spatial width of optical solitons, Spectral theory in the inverse problem

1. Introduction

In this paper we build up a model for spatial solitons in photorefractive crystals process using the inverse problem in scattering theory.

In 1953, [1] N. Levinson established some important relations between phase variations (first species discontinuities) of the asymptotically solutions and scattering potentials. R. Iost and R. Newton [2] built potential functions using S matrix associated to a differential equations system. V. A. Marcenko performed calculation of the potential energy depending on phase scattering waves [3]. Thus, it appears the inverse problem in scattering theory with V. A. Marcenko and Z. S. Agranovichy [4]. T. Regge [5] introduces notion of complex orbital moment. J. J. Loeffel [6] studies the connection between the inverse problem and scattering potential. In [7] R. Newton builds up a potential function using phase variations at constant energy. The same author [8] makes the connection between complex angular moment and the inverse problem at constant energy. P. Redmond [9] makes a few interesting remarks regarding the role of scattering matrix about inverse problem. P. Sabatier [10] creates a general algorithm for the inverse problem at constant energy. The same author [11] approaches an approximate inverse problem using interpolation formula with applications in determination of spin-orbit potentials.

B. M. Levitan and I. Sargya [12] and P. Sabatier [13] studied the role the spectral theory in the inverse problem. P. Sabatier [14] studied the asymptotical properties of the potentials in inverse problem, and B. M. Levitan [15] studied the inverse problem in scattering quantum theory at constant energy.

Years'70 are characterized by the effort made for solving of nonlinear differential equations using the inverse problem and for singular envelope solutions (solitons) description.

V. E. Zakharov and A. B. Shabat [16,17] develop methods for solving nonlinear wave equations using the inverse method in the scattering theory. The two authors together with J. Satsuma and N. Yajima [18] and C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura [19,20,21] improved the application of the inverse method in solving nonlinear equations getting singular envelope solutions (solitons). M. J. Ahlowitz and H. Segur [22, 23] studied the effect of losses on optical solitons width as well as transversal and longitudinal stability of the optical tridimensional solitons.

A formalism of the inverse method in solving nonlinear differential equations (of SBS type) and for the interpretation optical spatial solitons in photorefractive crystals will be presented in this paper.

2. Mathematical

It is considered tools the wave equation in the form [24]:

$$\Delta\Phi + k^2\Phi - V(r)\Phi = 0. \tag{1}$$

Choosing $V(r)$ with spherical symmetry we use the development:

$$\Phi(\vec{r}) = \frac{\varphi(r)}{\sqrt{r}} Y_m^l(\theta,\varphi). \tag{2}$$

The equation for radial component yields [24]:

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\cdot\frac{d\varphi}{dr} + k^2\varphi - \frac{\lambda^2}{r^2}\varphi - V(r)\cdot\varphi(r) = 0 \tag{3}$$

where $\lambda = l + 1/2\big|_{l=0,1,2,3,...;\lambda=0,\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2},...}$.

The function $\varphi = \varphi(r) \to \varphi(r \mid k,\lambda)$ is a regular solution in $r = 0$. We impose for $V(r)$ the conditions:

$$\int_0^1 r|V(r)|dr < \infty \text{ (finite value)} \tag{4}$$

$$\int_1^\infty |V(r)|dr < \infty \text{ (finite value)} \tag{5}$$

If the conditions considered (4) and (5) out it results:

$$\lim_{r\to\infty}\varphi(r\,|\,k,\lambda)=\sqrt{\frac{2}{\pi\cdot r}}A(k,\lambda)\cdot\sin\left[kr-\frac{\pi}{2}\left(\lambda-\frac{1}{2}\right)+\delta(k,\lambda)\right] \quad (6)$$

In the asymptotical expression of $\varphi(r)$ defined in (6) we use notations for:

$A(k,\lambda)\rightarrow$ the wave scattering amplitude

$\delta(k,\lambda)\rightarrow$ the wave scattering phase.

With these data, one defines the inverse problem in scattering theory: $V(r)$ will be determined knowing the spectral data which characterize the scattering ($A(k,\lambda)$, $\delta(k,\lambda)$.

It will be analyzed later the case in which k=1. So, the spectral data take the form:

$$\begin{aligned} A(1,\lambda)&=A(\lambda)\big|_{\lambda=l+\frac{1}{2};l=0,1,2,\dots}\\ \delta(1,\lambda)&=\delta(\lambda)\big|_{\lambda=l+\frac{1}{2};l=0,1,2,\dots}\end{aligned} \quad (7)$$

The wave equation (3) can be written in integral form (as proposed by Regge [5]):

$$\varphi(r,\lambda)=I_\lambda(r)+\int_0^r k(r,\rho)\cdot I_\lambda(\rho)\frac{d\rho}{\rho} \quad (8)$$

$I_\lambda(r)\rightarrow$ are the first species Bessel functions modified of (λ) order.

Using Regge integral solution (8) and wave equation (3), we build up the operators:

$$\begin{aligned} \hat{D}(r)&=r^2\left[\frac{d^2}{dr^2}+\frac{1}{r}\cdot\frac{d}{dr}+1-V(r)\right]\\ \hat{D}_0(\rho)&=\rho^2\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\cdot\frac{d}{d\rho}+1\right]\\ \hat{\Delta}_0(r)&=r^2\left[\frac{d^2}{dr^2}+\frac{1}{r}\cdot\frac{d}{dr}\right]\\ \hat{B}_\lambda&=r^2\left[\frac{d^2}{dr^2}+\frac{1}{r}\cdot\frac{d}{dr}+1-\frac{\lambda^2}{r^2}\right];\hat{B}_\lambda\cdot I_\lambda(r)=0\end{aligned} \quad (9)$$

The wave equation as an equation with eigenvalues is written on the form:

$$\hat{D}(r)\varphi(r,\lambda)=\lambda^2\varphi(r,\lambda) \quad . \quad (10)$$

The relations between operators are:

$$\begin{aligned} \hat{B}_\lambda(r)+\lambda^2&=\hat{D}_0(r)\\ \hat{D}(r)&=\hat{D}_0(r)-r^2\cdot V(r)\\ \left[\hat{B}_\lambda(r)-r^2\cdot V(r)\right]\varphi(r)&=0\end{aligned} \quad . \quad (11)$$

In the following will be presented the method of R. Newton of solving the inverse problem. We attach to the integral equation (8) and to the operators system (9) the equation with eigenvalues on the form [25]:

$$\begin{aligned} \hat{D}(r)\cdot K(r,\rho)&=\hat{D}_0(\rho)\cdot K(r,\rho)\\ V(r)&=\frac{2}{r}\frac{d}{dr}K(r,r)\end{aligned} \quad . \quad (12)$$

The method of R. Newton consists in solving the equation with eigenvalues:

$$\hat{D}_0(r)\cdot F(r,\rho)=\hat{\Delta}_0(\rho)\cdot F(r,\rho) \quad (13)$$

where $F(0,\rho)=F(r,0)=0$.

We decompose $F(r,\rho)$ under the form:

$$F(r,\rho)=\sum_{l=0}^{\infty}C_{l+\frac{1}{2}}\cdot I_{l+\frac{1}{2}}(r)\cdot I_{l+\frac{1}{2}}(\rho) \quad ,(14)$$

and thus we obtain an integral equation similar to that in (8), but for the integral nucleus $K(r,\rho)$:

$$\begin{aligned} K(r,\rho)&=F(r,\rho)+\int_0^r K(r,z)\cdot F(z,\rho)\cdot\frac{dz}{z}\Rightarrow\left\{\hat{D}(r)K(r,\rho)=\hat{D}_0(\rho)K(r,\rho)\right\}\\ \varphi(r,\lambda)&=I_\lambda(r)+\int_0^r K(r,\rho)\cdot I_\lambda(\rho)\cdot\frac{d\rho}{\rho}\Rightarrow\left\{\hat{D}(r)\varphi(r)=\lambda^2\varphi(r)\right\}\end{aligned} \quad . \quad (15)$$

By introducing the development (14) in integral equations (15), we obtain the integral nucleus expression $K(r,\rho)$ on the form [26]:

$$K(r,\rho)=\sum_{l=0}^{\infty}C_{l+\frac{1}{2}}\cdot I_{l+\frac{1}{2}}(\rho)\cdot\varphi(r,l+1/2).(16)$$

The potential function $V(r)$ of (12) takes the form [26,27]:

$$V(r)=\frac{2}{r}\sum_{l=0}^{\infty}C_{l+\frac{1}{2}}\left[\varphi\left(r,l+\frac{1}{2}\right)\frac{dI_{l+1/2}(r)}{dr}+I_{l+\frac{1}{2}}(r)\cdot\frac{d\varphi(r,l+1/2)}{dr}\right] \quad (17)$$

where:

$$\varphi\left(r,l+\frac{1}{2}\right)=I_{l+\frac{1}{2}}(r)+\sum_{l'=0}^{\infty}C_{l'+\frac{1}{2}}\cdot\varphi\left(r,l'+\frac{1}{2}\right)\cdot L_{ll'}(r) \quad (17')$$

where:

$$L_{ll'}(r)=\int_0^r I_{l+\frac{1}{2}}(\rho)\cdot I_{l'+\frac{1}{2}}(\rho)\cdot\frac{d\rho}{\rho}. \quad (18)$$

The algebraic equations system (17`) is solved in comparison with unknown $C_{l'+\frac{1}{2}}$, which later are introduced in (17) for the determination of the potential function $V(r)$.

One calculates the dependence of the coefficients $C_{l'+\frac{1}{2}}$ as a function of spectral data (phase_variation: $\delta(1, l+1/2)$)

The algebraic system (17`) is multiplied by $\sqrt{\frac{\pi r}{2}}$ and one gets the system:

$$\sqrt{\frac{\pi r}{2}}\varphi\left(r, l+\frac{1}{2}\right)=\sqrt{\frac{\pi r}{2}}\cdot I_{l+\frac{1}{2}}(r)+\sum_{l'=0}^{\infty}C_{l'+\frac{1}{2}}\cdot\sqrt{\frac{\pi r}{2}}\cdot\varphi\left(r, l'+\frac{1}{2}\right)\cdot L_{ll'}(r) \quad (19)$$

The equation system (18) cross to the limit $r \to \infty$; thus results:

$$\lim_{r\to\infty}\sqrt{\frac{\pi r}{2}}\cdot\varphi\left(r, l+\frac{1}{2}\right)=A\left(1, l+\frac{1}{2}\right)\cdot\sin\left[r-\frac{\pi l}{2}+\delta\left(1, l+\frac{1}{2}\right)\right] \quad (20)$$

Under the circumstances, the system becomes:

$$\begin{aligned}A\left(1, l+\frac{1}{2}\right)\cdot\sin\left[r-\frac{\pi l}{2}+\delta\left(1, l+\frac{1}{2}\right)\right]&=\lim_{r\to\infty}\sqrt{\frac{\pi r}{2}}\cdot I_{l+\frac{1}{2}}(r)+\\+\sum_{l'=0}^{\infty}C_{l'+\frac{1}{2}}\cdot A\left(1, l'+\frac{1}{2}\right)\cdot\sin\left[r-\frac{\pi l'}{2}+\delta\left(1, l'+\frac{1}{2}\right)\right]&\cdot\lim_{r\to\infty}L_{ll'}(r)\end{aligned} \quad (21)$$

We use asymptotic solutions for Bessel functions:

$$\begin{aligned}\lim_{r\to\infty}I_{l'+\frac{1}{2}}&=\sqrt{\frac{2}{\pi r}}\cdot\sin\left(r-\frac{\pi l'}{2}\right)\\ \lim_{r\to\infty}I_{l'-\frac{1}{2}}&=\sqrt{\frac{2}{\pi r}}\cdot\cos\left(r-\frac{\pi l'}{2}\right)\\ \lim_{r\to\infty}I_{l'+\frac{3}{2}}&=-\sqrt{\frac{2}{\pi r}}\cdot\cos\left(r-\frac{\pi l'}{2}\right)\end{aligned} \quad (22)$$

Thus, results:

$$\lim_{r\to\infty}L_{ll'}(r)=\frac{2}{\pi}\cdot\frac{\sin\left[\frac{\pi}{2}(l-l')\right]}{(l+l'+1)(l-l')}.$$

Introducing these relations in algebraic equation system (20) we obtain an algebraic unknown equation system $\left[C_{l'+\frac{1}{2}}\right]$ [28-30]:

$$\begin{aligned}&A\left(1,l+\frac{1}{2}\right)=e^{-i\delta\left(1,l+\frac{1}{2}\right)}+\\&+\frac{2}{\pi}\sum_{l'=0}^{\infty}\frac{C_{l'+\frac{1}{2}}\cdot A\left(1,l'+\frac{1}{2}\right)}{(l+l'+1)(l-l')}\sin\left[\frac{\pi}{2}(l-l')\right]\cdot e^{i\frac{\pi}{2}(l-l')+i\left\{\delta\left(1,l'+\frac{1}{2}\right)-\delta\left(1,l+\frac{1}{2}\right)\right\}}\end{aligned} \quad (23)$$

We observe that the asymptotic equation system (23) is no longer dependent on r, it only enssures the coefficients dependence $C_{l'+\frac{1}{2}}$ function of spectral data $\left\{A\left(1, l+\frac{1}{2}\right); \delta\left(1, l+\frac{1}{2}\right)\right\}$ therefore implicit and the structural dependence of the $V(r)$ potential function of spectral data.

We perform the spectral analysis of the following operators [31,32]:

$$\begin{aligned}\hat{\Delta}_0(r)&=r\cdot\frac{d}{dr}\left[r\cdot\frac{d}{dr}\right]\to(nondisturbed\ \ operator)\\ \hat{D}_0(r)&=r\cdot\frac{d}{dr}\left[r\cdot\frac{d}{dr}\right]+r^2\to(nondisturbed\ \ operator)\\ \hat{D}(r)&=r\cdot\frac{d}{dr}\left[r\cdot\frac{d}{dr}\right]+r^2-r^2V(r)\to(disturbed\ \ operator)\end{aligned}. \quad (24)$$

We use Fourier transforms in the form:

$$\begin{aligned}F(\tau)&=\int_{-\infty}^{+\infty}f(x)e^{-i\tau x}dx\\ f(x)&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\tau)e^{+i\tau x}d\tau\end{aligned}. \quad (25)$$

One makes the substitution $e^x = r$ and then the Fourier transforms takes the form:

$$\begin{aligned}F(\tau)&=\int_0^{\infty}g(r)\cdot r^{-i\tau}\frac{dr}{r}\Bigg|_{r\in[0,\infty)}\\ g(r)&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\tau)r^{+i\tau}d\tau\Bigg|_{\tau\in(-\infty,+\infty)}\end{aligned}. \quad (26)$$

The spectra associated to the operators (24) (i.e. the solutions) has the form:

$$\begin{aligned}&\hat{\Delta}_0\to\hat{\Delta}_0(r)\varphi_0(r)=\lambda^2\varphi_0(r);\ \varphi_0=\frac{r^{\lambda}}{2^{\lambda}\Gamma(1+\lambda)};\\ &\hat{D}_0\to\hat{D}_0(r)I_{\lambda}(r)=\lambda^2 I_{\lambda}(r);\\ &\hat{D}\to\hat{D}(r)\varphi(r)=\lambda^2\varphi(r),\end{aligned} \quad (27)$$

where $I_{\lambda}(r)$ are I species and I order – modified Bessel functions.

We use the crossing operators $\left\{\hat{X}_{\Delta_0,D_0}, \hat{X}_{\Delta_0,D}, \hat{X}_{D_0,D}\right\}$ defined as:

$$\begin{aligned}\hat{X}_{\Delta_0,D_0}&=1+\int_0^r A_{\Delta_0,D_0}(r,\rho)\cdot\frac{d\rho}{\rho}\\ \hat{X}_{\Delta_0,D}&=1+\int_0^r A_{\Delta_0,D}(r,\rho)\cdot\frac{d\rho}{\rho}\\ \hat{X}_{D_0,D}&=1+\int_0^r A_{D_0,D}(r,\rho)\cdot\frac{d\rho}{\rho}\end{aligned}. \quad (28)$$

The application method of the crossing operators [33-36]:

$$\hat{X}_{\Delta_0,D}:\ \varphi(r,\lambda)=\varphi_0(r,\lambda)+\int_0^r A_{\Delta_0,D_0}(r,\rho)\cdot\varphi_0(\rho,\lambda)\frac{d\rho}{\rho}$$

$$\hat{X}^{-1}_{\Delta_0,D}=\hat{X}_{D,\Delta_0}:\varphi_0(r,\lambda)=\varphi(r,\lambda)-\int_0^r \widetilde{A}_{\Delta_0,D}(r,\rho)\cdot\varphi(\rho,\lambda)\frac{d\rho}{\rho} \tag{29}$$

where:

$$\varphi_0(r,\lambda)=\frac{r^{\lambda}}{2^{\lambda}\cdot\Gamma(1+\lambda)}; \tag{30}$$

Using the representation (28) and (29), we build up the fundamental integral equation for solving the inverse problem (the integral Ghelfand-Levitan-Marcenko equation; GLM).

We use the integral equation:

$$\varphi(r,\lambda)=\varphi_0(r,\lambda)+\int_0^r A_{\Delta_0 D}(r,z)\cdot\varphi_0(z,\lambda)\frac{dz}{z}. \tag{31}$$

Given the case of discret spectrum $\lambda^2>0$; $\lambda\in R$; $\lambda\to\lambda_n\big|_{n\in N}$.

We multiply the integral equation of crossing operator with $\varphi_0(\rho,\lambda_n)$:

$$\varphi_0(\rho,\lambda_n)\cdot\varphi(r,\lambda_n)=\varphi_0(\rho,\lambda_n)\cdot\varphi_0(r,\lambda_n)+ \\ +\int_0^r A_{\Delta_0 D}(r,z)\cdot\varphi_0(\rho,\lambda_n)\cdot\varphi_0(z,\lambda_n)\frac{dz}{z} \tag{32}$$

We divide the relation (32) by a_n^2 and we count after n from 1 to infinite.

$$\sum_{n=1}^{\infty}\frac{1}{a_n^2}\cdot\varphi_0(\rho,\lambda_n)\varphi(r,\lambda_n)=\sum_{n=1}^{\infty}\frac{\varphi_0(\rho,\lambda_n)\cdot\varphi_0(r,\lambda_n)}{a_n^2}+ \\ +\int_0^r A_{\Delta_0 D}(r,z)\left\{\sum_{n=1}^{\infty}\frac{1}{a_n^2}\varphi_0(\rho,\lambda_{1n})\cdot\varphi_0(z,\lambda_n)\right\}\frac{dz}{z} \tag{33}$$

Given the case of negative spectrum (continuum spectrum): $\lambda^2<0\to\lambda=\pm i\tau$; $\tau\in R$. In this case $\lambda=+i\tau$.

We start as in the case of discrete spectrum with the crossing integral equation written in the form:

$$\varphi(r,i\tau)=\varphi_0(r,i\tau)+\int_0^r A_{\Delta_0 D}(r,z)\cdot\varphi_0(z,i\tau)\frac{dz}{z}. \tag{34}$$

We multiply (34) with the integral (equivalent Fourier transform): $\left(\int_{-\infty}^{+\infty}\frac{1}{2}\varphi_0(\rho,-i\tau)\cdot\frac{\tau\cdot d\tau}{sh(\pi\tau)}\right)$. The equation (34) becomes:

$$\frac{1}{2}\int_{-\infty}^{+\infty}\varphi_0(\rho,-i\tau)\cdot\varphi(r,i\tau)\frac{\tau\cdot d\tau}{sh(\pi\tau)}=\frac{1}{2}\int_{-\infty}^{+\infty}\varphi_0(\rho,-i\tau)\cdot\varphi_0(r,i\tau)\cdot\frac{\tau\cdot d\tau}{sh(\pi\tau)}+ \\ +\int_0^r A_{\Delta_0 D}(r,z)\cdot\left\{\frac{1}{2}\int_{-\infty}^{+\infty}\frac{1}{2}\varphi_0(\rho,-i\tau)\cdot\varphi_0(z,i\tau)\frac{\tau\cdot d\tau}{sh(\pi\tau)}\right\}\cdot\frac{dz}{z} \tag{35}$$

We use several integral representations of the form:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\rho^{-i\tau}r^{+i\tau}d\tau=r\cdot\delta(r-\rho)$$

$$\frac{1}{2}\int_{-\infty}^{+\infty}\varphi_0(r,i\tau)\cdot\varphi_0(\rho,-i\tau)\frac{\tau\cdot d\tau}{sh(\pi\tau)}=r\delta(r-\rho)$$

$$\sum_{n=1}^{\infty}\frac{\varphi(r,\lambda_n)\cdot\varphi(\rho,\lambda_n)}{\|\varphi(r,\lambda_n)\|^2}+\frac{1}{2}\int_{-\infty}^{+\infty}\left[\varphi(r,i\tau)-\frac{\mu(-i\tau)}{\mu(+i\tau)}\varphi(r,-i\tau)\right]\varphi(\rho,-i\tau)\frac{\tau\cdot d\tau}{sh(\pi\tau)}=r\delta(r-\rho) \tag{36}$$

where: $a_n^2=\|\varphi(r,\lambda_n)\|^2$;

Thus, for $\rho<r$ and $\lambda=+i\tau$ it results:

$$A_{\Delta_0 D}(r,\rho)=\frac{1}{2}\int_{-\infty}^{+\infty}\varphi_0(\rho,-i\tau)\cdot\varphi(r,i\tau)\frac{\tau\cdot d\tau}{sh(\pi\tau)}. \tag{37}$$

And for $\lambda=-i\tau$ it results:

$$\varphi(r,-i\tau)=\varphi_0(r,-i\tau)+\int_0^r A_{\Delta_0 D}(r,z)\cdot\varphi_0(z,-i\tau)\frac{dz}{z}. \tag{38}$$

Using the same algorithm for the continuum spectrum as for the discrete one, the integral equation is written in the form:

$$A_{\Delta_0 D}(r,\rho)+F_{\Delta_0 D}(r,\rho)+\int_0^r A_{\Delta_0 D}(r,z)\cdot F_{\Delta_0 D}(z,\rho)\frac{dz}{z}=0 \tag{39}$$

where:

$$F_{\Delta_0 D}(r,\rho)=\sum_{n=1}^{\infty}\frac{1}{a_n^2}\varphi_0(\rho,\lambda_n)\cdot\varphi_0(r,\lambda_n)- \\ -\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\mu(-i\tau)}{\mu(+i\tau)}\varphi_0(\rho,-i\tau)\cdot\varphi_0(r,-i\tau)\frac{\tau\cdot d\tau}{sh(\pi\tau)} \tag{40}$$

the first term representing the contribution of discrete spectrum, and the second term the contribution of the continuum spectrum.

In the expression (40), a_n^2 and $\frac{\mu(-i\tau)}{\mu(+i\tau)}$ contain the spectral data induced by $V(r)$.

Calculation example: we choose

$$\varphi_0(r,\lambda_n)=\frac{\left(\frac{r}{2}\right)^{\lambda_n}}{\Gamma(1+\lambda_n)};$$

The function $F_{\Delta_0 D}(r,\rho)$ becomes $f_{\Delta_0 D}(r\cdot\rho)$ (function of product $r\cdot\rho$) and yields:

$$f_{\Delta_0 D}(r\cdot\rho)=\sum_{n=1}^{\infty}\frac{\left(\frac{\rho\cdot r}{4}\right)^{\lambda_n}}{a_n^2\cdot\Gamma^2(1+\lambda_n)}-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\mu(-i\tau)}{\mu(i\tau)}\cdot\frac{\left(\frac{\rho\cdot r}{4}\right)^{-i\tau}}{\Gamma^2(1-i\tau)}\cdot\frac{\tau d\tau}{sh(\pi\tau)}. \tag{41}$$

The function $f_{\Delta_0 D}(r\cdot\rho)$ is named also *the function Regge-Loeffel* [6]. The integral equation (GLM) has the form:

$$A_{\Delta_0 D}(r,\rho)+f_{\Delta_0 D}(r\cdot\rho)+\int_0^r A_{\Delta_0 D}(r,z)\cdot f_{\Delta_0 D}(z\cdot\rho)\frac{dz}{z}=0 \qquad (42)$$

We choose a crossing operator $\hat{X}_{\Delta_0 D_0}$ with the form:

$$\hat{X}_{\Delta_0 D_0}\varphi_0(r,\lambda)=I_\lambda(r) \qquad (43)$$

The associated integral equation has the form:

$$I_\lambda(r)=\varphi_0(r,\lambda)+\int_0^r A_{\Delta_0 D_0}(r,z)\cdot\varphi_0(z,\lambda)\frac{dz}{z}. \qquad (44)$$

We build up the spectrum of the operator $\hat{\Delta}_0$, based on the eigenvalues equation:

$$\hat{\Delta}_0(r)\varphi_0(r,\lambda)=\lambda^2\varphi_0(r,\lambda);\quad \varphi_0(r,\lambda)=\frac{r^\lambda}{2^\lambda\cdot\Gamma(1+\lambda)}. \qquad (45)$$

The operator $\hat{\Delta}_0(r)$ has not discrete spectrum,

$$\text{so:}\ A(\lambda)=0,\quad \delta(\lambda)=0,\quad \frac{\mu(-i\tau)}{\mu(+i\tau)}=-1 \qquad (46)$$

Therefore, the continuum spectrum of the operator $\hat{\Delta}_0$ takes the form:

$$r\cdot\delta(r-\delta)=+\frac{1}{2}\int_{-\infty}^{+\infty}\varphi_0(r,i\tau)\cdot\varphi_0^*(\rho,i\tau)\frac{\tau\cdot d\tau}{sh(\pi\tau)}. \qquad (47)$$

We calculate the spectrum of the operator $\hat{D}_0(r)$; the eigen values equation is:

$$\hat{D}_0(r)\varphi(r,\lambda)=\lambda^2\varphi(r,\lambda), \qquad (48)$$

where $\varphi(r,\lambda)=I_\lambda(r)$.

The spectral equation attached to operator $\hat{D}_0(r)$ has the form:

$$r\cdot\delta(r-\rho)=\sum_{n=1}^{\infty}\frac{I_{\lambda_n}(\rho)I_{\lambda_n}(r)}{\left\|I_{\lambda_n}\right\|^2}+\frac{1}{2}\int_{-\infty}^{+\infty}\left[I_{i\tau}(r)-\frac{\mu(-i\tau)}{\mu(+i\tau)}\cdot I_{-i\tau}(r)\right]\cdot I_{-i\tau}(\rho)\frac{\tau\cdot d\tau}{sh(\pi\tau)} \qquad (49)$$

The discrete spectrum norm has the form:

$$\left\|I_{\lambda_n}\right\|^2\overset{Def}{=}\int_0^\infty I_{\lambda_n}^2(r)\cdot\frac{dr}{r}=\frac{1}{2\lambda_n}. \qquad (50)$$

For the spectral quantities calculation $\mu(\pm i\tau)$, will take into account that the Bessel functions $I_\lambda(r)$ accept asymptotic solutions on the form:

$$I_\lambda(r)=\sqrt{\frac{2}{\pi r}}\sin\left[r+\frac{\pi}{2}\left(\lambda-\frac{1}{2}\right)\right],\quad \frac{dI_\lambda(r)}{dr}=\sqrt{\frac{2}{\pi r}}\cos\left[r+\frac{\pi}{2}\left(\lambda-\frac{1}{2}\right)\right]. \qquad (51)$$

We identify solutions (50) with general asymptotic solutions which include spectral data $A(\lambda)$, $\delta(\lambda)$ described in (5) and (6) and which can be written in the form:

$$\varphi(r,\lambda)=\sqrt{\frac{2}{\pi r}}\cdot A(\lambda)\cdot\sin\left[r+\delta(r)-\frac{\pi}{2}\left(\lambda-\frac{1}{2}\right)\right],\quad \varphi'(r,\lambda)=\sqrt{\frac{2}{\pi r}}\cdot A(\lambda)\cdot\cos\left[r+\delta(r)-\frac{\pi}{2}\left(\lambda-\frac{1}{2}\right)\right]. \qquad (52)$$

As a result of identification we obtained:

$$A(\lambda)=1,\quad \delta(\lambda)=0,\quad \mu(i\tau)=\sin\left(x_0+\frac{\pi}{2}i\tau\right),\quad \mu(-i\tau)=\sin\left(x_0-\frac{\pi}{2}i\tau\right),\quad \lambda_n=2n,\quad \lim_{x_0\to 0}\frac{\mu(-i\tau)}{\mu(+i\tau)}=-1 \qquad (53)$$

So, the symbolic representation of the spectrum of $\hat{D}_0(r)$ takes the form:

$$r\cdot\delta(r-\rho)=\sum_{n=1}^{\infty}4n\cdot I_{2n}(\rho)\cdot I_{2n}(r)+\frac{1}{2}\int_{-\infty}^{+\infty}\left[I_{i\tau}(r)+I_{-i\tau}(r)\right]\cdot I_{-i\tau}(\rho)\frac{\tau\cdot d\tau}{sh(\pi\tau)}. \qquad (54)$$

The integral equation of scattering from (38) can be written in the new conditions:

$$-A_{\Delta_0 D_0}(r,\rho)=F_{\Delta_0 D_0}(r,\rho)+\int_0^r A_{\Delta_0 D_0}(r,z)\cdot F_{\Delta_0 D_0}(z,\rho)\frac{dz}{z} \qquad (55)$$

or otherwise written (in terms of the crossing operator $\hat{X}_{\Delta_0 D_0}$) in the form:

$$-A_{\Delta_0 D_0}(r,\rho)=\hat{X}_{\Delta_0 D_0}\cdot F_{\Delta_0 D_0}=F_{\Delta_0 D_0}(r,\rho)+\int_0^r A_{\Delta_0 D_0}(r,z)\cdot F_{\Delta_0 D_0}(z,\rho)\frac{dz}{z} \tag{56}$$

Thus, the integral equation of scattering can be written as:

$$-A_{\Delta_0 D_0}(r,\rho)=\hat{X}_{\Delta_0 D_0}(r)\cdot F_{\Delta_0 D_0}. \tag{57}$$

From the relation (57) we write the inverse relation:

$$F_{\Delta_0 D_0}=-\hat{X}^{-1}_{\Delta_0 D_0}\cdot A_{\Delta_0 D_0}(r,\rho). \tag{58}$$

But from the wave equation it results:

$$\hat{\Delta}_0(r)\cdot A_{\Delta_0 D_0}=\hat{\Delta}_0(r)\cdot A_{\Delta_0 D_0}. \tag{59}$$

The solution of equation (59) is just the transformation nucleus, $A_{\Delta_0 D_0}$, that has the form:

$$A_{\Delta_0 D_0}(r,\rho)=-\sum_{k=1}^{\infty}\frac{1}{\Gamma(k)}\cdot I_k(r)\cdot\left(\frac{\rho}{2}\right)^k \tag{60}$$

Also, it results:

$$F_{\Delta_0 D_0}(r\cdot\rho)=\frac{1}{2}\cdot i\sqrt{\rho r}\cdot I_1\left(i\sqrt{r\rho}\right) \text{ (the Regge-Loeffel function).} \tag{61}$$

In conclusion, we mention that in the integral equation of scattering:

$$A_{\Delta_0 D_0}(r,\rho)+f_{\Delta_0 D_0}(r\cdot\rho)+\int_0^r A_{\Delta_0 D_0}(r,z)\cdot f_{\Delta_0 D_0}(z\cdot\rho)\frac{dz}{z}=0, \tag{62}$$

the known function (the input quantity) $f_{\Delta_0 D_0}(r\cdot\rho)$ contains spectral data of the nonlinear operator; thus:

$$f_{\Delta_0 D_0}(r\cdot\rho)=\sum_{n=1}^{\infty}\frac{\left(\frac{\rho\cdot r}{4}\right)^{\lambda_n}}{a_n^2\cdot\Gamma(1+\lambda_n)}-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\mu(-i\tau)}{\mu(+i\tau)}\cdot\frac{\left(\frac{\rho\cdot r}{4}\right)^{-i\tau}}{\Gamma^2(1-i\tau)}\cdot\frac{\tau\cdot d\tau}{sh(\pi\tau)} \tag{63}$$

and the solution of integral equation (GLM) is the integral nucleus function $A_{\Delta_0 D_0}$.

Finally will be represented a spectral development:

Given the operator $\hat{D}(r)$ and the wave equation (equation with eigen values): $\hat{D}(r)\varphi(r,\lambda)=\lambda^2\varphi(r,\lambda)$; the discrete spectrum is represented by the eigenvalues: $\lambda^2>0\rightarrow(\lambda_n)$; the continuous spectrum: $\lambda^2<0$; $\lambda=\pm i\tau$; the symbolic form of the spectrum of the operator $\hat{D}(r)$ is:

$$r\delta(r-\rho)=\sum_{n=1}^{\infty}\frac{\varphi(\rho,\lambda_n)\cdot\varphi(r,\lambda_n)}{\left\|\varphi(r,\lambda_n)\right\|^2}+\frac{1}{2}\int_{-\infty}^{+\infty}\left[\varphi(r,i\tau)-\frac{\mu(-i\tau)}{\mu(+i\tau)}\varphi(r,-i\tau)\right]\varphi(\rho,-i\tau)\frac{\tau d\tau}{sh(\pi\tau)}. \tag{64}$$

The spectral data (Fourier transform after ρ) have the form:

$$\left\|\varphi(r,\lambda_n)\right\|^2\overset{Def}{=}\int_0^{\infty}\varphi^2(r,\lambda_n)\frac{dr}{r}=\frac{A^2(\lambda_n)}{\pi\lambda_n}\cdot\left[\frac{\pi}{2}-\delta'(\lambda_n)\right];$$
$$\delta'(\lambda_n)=\left(\frac{d\delta}{dr}\right)_{\lambda_n} \tag{65}$$

$$\mu(i\tau)=\mu(i\tau;x_0)=A(-i\tau)\cdot\sin\left[x_0+\delta(-i\tau)+\frac{\pi}{2}i\tau\right], \tag{66}$$

where $A(\lambda)$ and $\delta(\lambda)$ are the spectral data from the asymptotic solutions of the wave equation.

These asymptotic solutions have the form:

$$\sqrt{\frac{\pi r}{2}}\cdot\varphi(r,\lambda)\cong A(\lambda)\cdot\sin\left[r+\delta(\lambda)-\frac{\pi}{2}\left(\lambda-\frac{1}{2}\right)\right]$$
$$\sqrt{\frac{\pi r}{2}}\cdot\varphi'(r,\lambda)\cong A(\lambda)\cdot\cos\left[r+\delta(\lambda)-\frac{\pi}{2}\left(\lambda-\frac{1}{2}\right)\right]$$
$$;\ \varphi'=\left(\frac{d\varphi}{dr}\right)_{\lambda_n}. \tag{67}$$

3. The qualitative analysis of the nonlinear equations using integral relations for solitons in photorefractive nonlinear crystals

The constitutive equations concerning the optical envelope dynamic (Φ_{x_1}, Φ_{y_1}) have the form [31,32]:

$$2i\frac{\partial\Phi_{x_1}}{\partial\eta'}+\frac{\partial^2\Phi_{x_1}}{\partial\rho^2}+\mu\cdot NL(\eta',\rho)\Phi_{x_1}=0$$
$$2i\frac{\partial\Phi_{y_1}}{\partial\eta'}+\frac{\partial^2\Phi_{y_1}}{\partial\rho^2}+NL(\eta',\rho)\Phi_{y_1}=0 \tag{68}$$

where μ represent an asymmetry coefficient, and the nonlinear component is defined by:

$$NL(\eta',\rho)=\frac{1}{1+\frac{2r}{\pi}\gamma^2(z)\cdot\left[\Phi_{x_1}^2+\Phi_{y_1}^2\right]}, \tag{69}$$

where $r=\frac{I_0}{I_B}$ is the ratio between the maximum intensity and background intensity, and

$$\gamma=\frac{e^{-\frac{\alpha}{2}\cdot z}}{\left|\cos\left[\frac{1}{2}\left(g_1k\cdot z-\frac{\pi}{2}+\varphi\right)\right]\right|}, \tag{70}$$

where α represent an linear loss coefficient, and $g_1 \cdot k$ nonlinear spatial frequency.

Given the Fourier transform operators defined in the form:

$$\hat{\tilde{F}} = \int_{-\infty}^{+\infty} e^{-2\pi ik\rho} d\rho \\ \hat{\tilde{F}}^{-1} = \int_{-\infty}^{+\infty} e^{+2\pi ik\rho} dk \quad . \tag{71}$$

and given the Fourier transforms:

$$\tilde{\Phi}_{x_1}(\eta', k) = \int_{-\infty}^{+\infty} \Phi_{x_1}(\eta', \rho) \cdot e^{-2\pi ik\rho} d\rho \\ \tilde{\Phi}_{y_1}(\eta', k) = \int_{-\infty}^{+\infty} \Phi_{y_1}(\eta', \rho) \cdot e^{-2\pi ik\rho} d\rho \quad . \tag{72}$$

and given the inverse Fourier transforms:

$$\Phi_{x_1}(\eta', \rho) = \int_{-\infty}^{+\infty} \hat{\Phi}_{x_1}(\eta', k) \cdot e^{+2\pi ik\rho} dk \\ \Phi_{y_1}(\eta', \rho) = \int_{-\infty}^{+\infty} \hat{\Phi}_{y_1}(\eta', k) \cdot e^{+2\pi ik\rho} dk \quad . \tag{73}$$

Finishing the preliminaries, we pass to the effective calculation stages:

We perform the Fourier transforms of the (68) and we obtain:

$$2i \frac{\partial \tilde{\Phi}_{x_1}(\eta', k)}{\partial \eta'} + (-2\pi ik)^2 \tilde{\Phi}_{x_1}(\eta', k) + \mu \int_{-\infty}^{+\infty} NL(\eta', \rho) \cdot \Phi_{x_1}(\eta', \rho) \cdot e^{-2\pi ik\rho} d\rho = 0 \\ 2i \frac{\partial \tilde{\Phi}_{y_1}(\eta', k)}{\partial \eta'} + (-2\pi ik)^2 \tilde{\Phi}_{y_1}(\eta', k) + \int_{-\infty}^{+\infty} NL(\eta', \rho) \cdot \Phi_{y_1}(\eta', \rho) \cdot e^{-2\pi ik\rho} d\rho = 0 \quad . \tag{74}$$

we process the convolution integrals from (74) and we obtain:

$$2i \frac{\partial \tilde{\Phi}_{x_1}(\eta', k)}{\partial \eta'} - 4\pi^2 k^2 \tilde{\Phi}_{x_1}(\eta', k) + \mu \int_{-\infty}^{+\infty} \tilde{N}L(\eta', k - k') \cdot \tilde{\Phi}_{x_1}(\eta', k') dk' = 0 \\ 2i \frac{\partial \tilde{\Phi}_{y_1}(\eta', k)}{\partial \eta'} - 4\pi^2 k^2 \tilde{\Phi}_{y_1}(\eta', k) + \int_{-\infty}^{+\infty} \tilde{N}L(\eta', k - k') \cdot \Phi_{y_1}(\eta', k') dk' = 0 \tag{75}$$

Thus, from the point of view of the inverse problem, $\tilde{\Phi}_{x_1}(\eta', k)$ and $\tilde{\Phi}_{y_1}(\eta', k)$ represent the spectral data of the problem. The equations (74) integrated upon η' represent the time and space evolution of the spectral data associated to the inverse problem. As a result of the integration upon η' we obtain the equations [33-36]:

$$\tilde{\Phi}_{x_1}(\eta', k) = e^{-2i\pi^2\eta' k^2} \left\{ C_1(k) - \frac{\mu}{2i} \int_0^{\eta'} d\eta'' \int_{-\infty}^{+\infty} \tilde{N}L(\eta'', k - k') \cdot \tilde{\Phi}_{x_1}(\eta'', k') e^{+2i\pi^2 k^2 \eta''} dk' \right\} \\ \tilde{\Phi}_{y_1}(\eta', k) = e^{-2i\pi^2\eta' k^2} \left\{ C_2(k) - \frac{1}{2i} \int_0^{\eta'} d\eta'' \int_{-\infty}^{+\infty} \tilde{N}L(\eta'', k - k') \cdot \tilde{\Phi}_{y_1}(\eta'', k') e^{+2i\pi^2 k^2 \eta''} dk' \right\} \tag{76}$$

where the initial (spectral) conditions $C_1(k)$, $C_2(k)$ have the form:

$$C_1(k) = \tilde{\Phi}_{x_1}(0, k) = \int_{-\infty}^{+\infty} \Phi_{x_1}(0, \rho) \cdot e^{-2\pi ik\rho} d\rho \\ C_2(k) = \tilde{\Phi}_{y_1}(0, k) = \int_{-\infty}^{+\infty} \Phi_{y_1}(0, \rho) \cdot e^{-2\pi ik\rho} d\rho \quad , \tag{77}$$

where $\Phi_{x_1}(0, \rho)$ and $\Phi_{y_1}(0, \rho)$ are the initial data of the problem.

One operates the inverse transform of the field envelopes (76) and one gets (integral solutions in the form) [37,38]:

$$\Phi_{x_1}(\eta', \rho) = \Phi_{x_{1_0}}(\eta', \rho) - \frac{\mu}{2i} \int_0^{\eta'} d\eta'' \int_{-\infty}^{+\infty} \frac{\Phi_{x_1}(\eta'', \rho')}{1 + \frac{2r}{\pi}\gamma^2 \left[\Phi_{x_1}^2(\eta'', \rho') + \Phi_{y_1}^2(\eta'', \rho')\right]} \frac{e^{-\frac{(\rho-\rho')^2}{2i(\eta'-\eta'')}}}{\sqrt{2\pi i(\eta' - \eta'')}} d\rho' \\ \Phi_{y_1}(\eta', \rho) = \Phi_{y_{1_0}}(\eta', \rho) - \frac{1}{2i} \int_0^{\eta'} d\eta'' \int_{-\infty}^{+\infty} \frac{\Phi_{y_1}(\eta'', \rho')}{1 + \frac{2r}{\pi}\gamma^2 \left[\Phi_{x_1}^2(\eta'', \rho') + \Phi_{y_1}^2(\eta'', \rho')\right]} \frac{e^{-\frac{(\rho-\rho')^2}{2i(\eta'-\eta'')}}}{\sqrt{2\pi i(\eta' - \eta'')}} d\rho' \tag{77`}$$

where:

$$\Phi_{x_{1_0}}(\eta', \rho) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(\rho-\rho')^2}{2i\eta'}}}{\sqrt{2\pi i\eta'}} \cdot \Phi_{x_1}(0, \rho') d\rho' \\ \Phi_{y_{1_0}}(\eta', \rho) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(\rho-\rho')^2}{2i\eta'}}}{\sqrt{2\pi i\eta'}} \cdot \Phi_{y_1}(0, \rho') d\rho' \quad . \tag{78}$$

The initial conditions are contained (are supplied) in the equations (78). Thus:

$$\lim_{\eta' \to 0} \Phi_{x_1}(\eta', \rho') = \lim_{\eta' \to 0} \Phi_{x_{1_0}}(\eta', \rho') = \Phi_{x_1}(0, \rho') \\ \lim_{\eta' \to 0} \Phi_{y_1}(\eta', \rho') = \lim_{\eta' \to 0} \Phi_{y_{1_0}}(\eta', \rho') = \Phi_{y_1}(0, \rho') \quad . \tag{79}$$

We define the function:

$$NL(\eta'', \rho') \overset{Def}{=} \frac{1}{1 + \frac{2r}{\pi} \cdot \gamma^2 \left[\Phi_{x_1}^2(\eta'', \rho') + \Phi_{y_1}^2(\eta'', \rho')\right]} \quad . \tag{80}$$

<u>Exemple of calculation</u>:

We assume that the pump is accomplished by a Gaussian beam, so that we will have:

$$\Phi_{x_1}(0, \rho') = \frac{1}{\sqrt{\pi} \cdot \sigma_x} \cdot e^{-\frac{\rho'^2}{\sigma_x^2}} \\ \Phi_{y_1}(0, \rho') = \frac{1}{\sqrt{\pi} \cdot \sigma_y} \cdot e^{-\frac{\rho'^2}{\sigma_y^2}} \quad , \tag{81}$$

where: $\sigma_{x,y} = \mathrm{Re}(\sigma_{x,y}) + i \cdot \mathrm{Im}(\sigma_{x,y})$, and $\sigma_{x,y}$ belong to the complex numbers class ($\sigma_{x,y} \in C$).

We will have:

$$\Phi_{x_{1_0},y_{1_0}} = \frac{1}{\sqrt{\pi}\sigma_{x,y}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{(\rho-\rho')^2}{2i\eta'} - \frac{\rho'^2}{\sigma_{x,y}^2}}}{\sqrt{2\pi i\eta'}} d\rho' . \quad (82)$$

In these conditions, after integrating (81) we obtain:

$$\begin{aligned} \Phi_{x_{1_0}}(\eta',\rho) &= \frac{e^{-\frac{\rho^2}{\sigma_x^2+2i\eta'}}}{\sqrt{\pi(\sigma_x^2+2i\eta')}} \\ \Phi_{y_{1_0}}(\eta',\rho) &= \frac{e^{-\frac{\rho^2}{\sigma_y^2+2i\eta'}}}{\sqrt{\pi(\sigma_y^2+2i\eta')}} \end{aligned} . \quad (83)$$

We use the case of the circular symmetry ($\sigma_x = \sigma_y = \sigma_L = \sigma_{LR} + i\sigma_{LI}$). Thus, we will have:

$$\begin{aligned} \Phi_{x_1}(0,\rho') &= \Phi_{y_1}(0,\rho') = \frac{1}{\sqrt{\pi}\cdot\sigma_L}\cdot e^{-\frac{\rho^2}{\sigma_L^2}} \\ \Phi_{x_{1_0}}(\eta',\rho') &= \Phi_{y_{1_0}}(\eta',\rho') = \frac{e^{-\frac{\rho'^2}{\sigma_L^2+2i\eta'}}}{\sqrt{\pi(\sigma_L^2+2i\eta')}} \end{aligned} . \quad (84)$$

In the integral equations, the nonlinear function (nonlinear nucleus) and only that is written in the form:

$$NL(\eta'',\rho') = \frac{1}{1+\frac{4r}{\pi}\cdot\gamma^2\cdot\Phi_{x_{1_0}}^2(\eta'',\rho')} . \quad (85)$$

From the symmetry of the optical field envelopes it results:

$$\Phi_{y_1}(\eta',\rho) = \Phi_{x_1}(\eta',\rho)\Big|_{\mu=+1} . \quad (86)$$

So, from now on, we will effect the calculus on Φ_{x_1}, and $\Phi_{y_1} = \Phi_{x_1}\big|_{\mu=+1}$. Thus, the expression of Φ_{x_1} yields:

$$\Phi_{x_1} = \Phi_{x_{1_0}}(\eta',\rho) - \frac{\mu}{2i}\cdot D_x , \quad (87)$$

where:

$$D_x = \int_0^{\eta'} d\eta'' \int_{-\infty}^{+\infty} \frac{\Phi_{x_1}(\eta'',\rho')}{1+\left(\sqrt{\frac{4r}{\pi}}\gamma^2\right)^2\cdot\Phi_{x_{1_0}}^2(\eta'',\rho')}\cdot\frac{e^{-\frac{(\rho-\rho')^2}{2i(\eta'-\eta'')}}}{\sqrt{2\pi i(\eta'-\eta'')}} d\rho' . \quad (88)$$

In the first species approximation, in the D_x expression, we use the asymptotic representation to the limit:

$$\lim_{\Phi_{x_1}\to\Phi_{x_{1_0}}} D_x = D_{x_0} , \quad (89)$$

where $\Phi_{x_{1_0}}$ and D_{x_0} are asymptotic forms for Φ_{x_1} and D_x. Thus, for D_{x_0} it results the expression:

$$D_{x_0} = \int_0^{\eta'} d\eta'' \int_{-\infty}^{+\infty} \frac{\Phi_{x_{1_0}}(\eta'',\rho')}{\left[1+\left(\sqrt{\frac{4r}{\pi}}\gamma^2\right)^2\cdot\Phi_{x_{1_0}}^2(\eta'',\rho')\right]}\cdot\frac{e^{-\frac{(\rho-\rho')^2}{2i(\eta'-\eta'')}}}{\sqrt{2\pi i(\eta'-\eta'')}} d\rho' \quad (90)$$

Using the approximation:

$$\int_0^{\eta'} d\eta'' \to \eta' \quad (91)$$

and crossing to the limit $\eta'' \to \eta'$ result the algebraic form of D_{x_0}:

$$D_{x_0} = \frac{1}{4}\sqrt{\frac{\pi}{r}}\cdot\frac{\eta'}{\gamma(\eta')}\cdot\frac{1}{ch\left\{-\frac{\rho^2}{\sigma_L^2+2i\eta'} + \ln\left[\frac{\sqrt{\frac{4}{\pi}r\gamma^2}}{\sqrt{\pi(\sigma_l^2+2i\eta')}}\right]\right\}} \quad (92)$$

Thus, it results that the envelope solutions are corresponding each to another:

$$\begin{aligned} \Phi_{x_1}(\eta',\rho) &= \Phi_{x_{1_0}}(\eta',\rho) - \frac{\mu}{2i}\cdot D_{x_0} \\ \Phi_{y_1}(\eta',\rho) &= \Phi_{x_{1_0}}(\eta',\rho) - \frac{1}{2i}\cdot D_{x_0} \end{aligned} . \quad (93)$$

We enumerate a few properties of the envelope amplitudes:

$$\begin{aligned} \lim_{r\cdot\gamma^2\to 0} \Phi_{x_1}(\eta',\rho) &= \Phi_{x_{1_0}}(\eta',\rho)\cdot\left[1-\frac{\mu}{2i}\eta'\right] \\ \lim_{r\cdot\gamma^2\to 0} \Phi_{y_1}(\eta',\rho) &= \Phi_{x_{1_0}}(\eta',\rho)\cdot\left[1-\frac{1}{2i}\eta'\right] \end{aligned} , \quad (94)$$

$$\begin{aligned} \lim_{r\cdot\gamma^2\to\infty} \Phi_{x_1}(\eta',\rho) &= \Phi_{x_{1_0}}(\eta',\rho) \\ \lim_{r\cdot\gamma^2\to\infty} \Phi_{y_1}(\eta',\rho) &= \Phi_{x_{1_0}}(\eta',\rho) \end{aligned} . \quad (95)$$

The pure solitonic solution condition (as additive term and σ_L - complex) results under the form:

$$\gamma(\eta') = \frac{\pi\sqrt{\sigma_L^2+2i\eta'}}{4r} , \quad (96)$$

and the D_{x_0} expression becomes [36,37]:

$$D_{x_0} = \frac{\sqrt{\pi}\cdot\eta'}{4\sqrt{r}\cdot\gamma(\eta')}\cdot\frac{1}{\cosh\left[\left(\frac{\rho}{\sqrt{\frac{4r\gamma^2}{\pi^2}}}\right)^2\right]}. \quad (97)$$

The soliton width, $\Delta\rho$, will be defined as:

$$\Delta\rho = \frac{2}{\pi}\gamma(\eta')\cdot\sqrt{r}. \quad (98)$$

But, from the link relation (96) and the relation (98) it results from the initial condition for $\gamma(\eta')$:

$$\gamma(\eta'=0) = \frac{\pi}{2r}\cdot\sigma_L \qquad \text{(with } \sigma_L \text{ real)} \quad (99)$$

for the function:

$$D_{x_0} = \frac{\eta'}{2\sqrt{\pi}\Delta\rho}\cdot\frac{1}{\cosh\left[\left(\frac{\rho}{\Delta\rho}\right)^2\right]}. \quad (100)$$

If we calculate the optical soliton width, $\Delta\rho$, at ½ of maximum amplitude, it results [39,40]:

$$\Delta\rho = A\cdot\left[\left(\sqrt{B}-\sqrt{C}\right)+\frac{1}{4\sqrt{B}}\cdot\ln r+\frac{1}{4\sqrt{C}}\cdot\ln\frac{1}{r}\right]. \quad (101)$$

where:

$$A = \left|\sqrt{\sigma_L^2+2i\eta'}\right|;\; B = \ln\left[\gamma\frac{2\left(2+\sqrt{3}\right)}{\pi\cdot A}\right];$$

$$C = \ln\left[\gamma\frac{2\left(2-\sqrt{3}\right)}{\pi\cdot A}\right]. \quad (102)$$

If we linearize the expression of $\Delta\rho$ from (100), comes out the dependence:

$$\Delta\rho = \frac{A}{2\sqrt{B}}\cdot r+\frac{A}{2\sqrt{C}}\cdot\frac{1}{r}+A\left[\sqrt{B}-\sqrt{C}-\frac{3}{8}\frac{\sqrt{B}+\sqrt{C}}{\sqrt{BC}}\right] \quad (103)$$

wherefrom results an important quantity for the minimal soliton width. Thus:

$$\Delta\rho(r=1) = \frac{A}{8}\left\{\frac{B+C+2\sqrt{BC}\left[1+4(B-C)\right]}{B\cdot\sqrt{C}+\sqrt{B}\cdot C}\right\} \quad (104)$$

and

$$\Delta\rho_{\min} = A\cdot\left[\frac{1}{\sqrt[4]{BC}}+\left(\sqrt{B}-\sqrt{C}\right)-\frac{3}{8}\frac{\sqrt{B}+\sqrt{C}}{\sqrt{BC}}\right]. \quad (105)$$

The dependence $\Delta\rho(r)$ is presented in the Fig. 1.

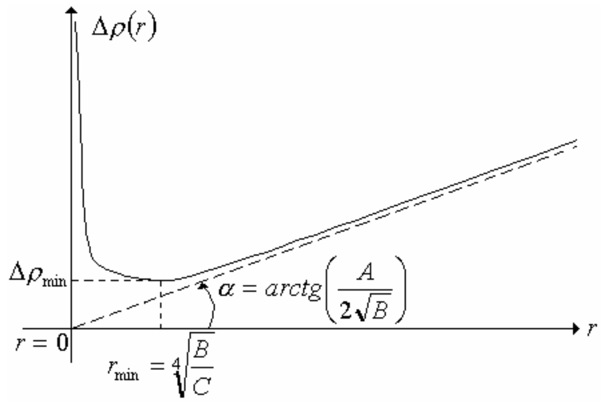

Fig. 1. The dependence of the optical soliton width depending on the parameter $r = I_0 / I_B$.

4. Conclusions

In this paper was presented the spectral theory of the nonlinear operators with applications to the inverse scattering theory (the Ghelfand-Levitan-Marcenko theory) concerning the screening spatial optical solitons theory. Analytically, there was evaluated the strength of the screening of spatial solitons in photorefractive crystals, using specific elements from the inverse problem of the scattering theory (at constant energy) after P. Sabatier.

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