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Energy density of the cubic and spherical cavities with low adiabatic invariant

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The Planck radiation spectrum of ideal cubic and spherical cavities with low adiabatic invariants, $\gamma = TV^{1/3}$, is discrete and strongly dependent on the cavity geometry and temperature. This behavior is the consequence of the random distribution of the state weights in the cubic cavity and of the random overlapping of the successive multiplet components, in the case of spherical cavity. The total energy density of cavities with low adiabatic invariant, γ (obtained by summing up the exact contributions of the eigenvalues and their weights) does not obey any longer Stefan-Boltzmann law. The new law includes a corrective factor depending on γ and imposes an exponential-type decrease of the total energy density to zero, when $\gamma \to 0$. This special quantum regime, defined by limits of principal quantum number or of adiabatic invariant, appears to be similar for cubic and spherical cavities. The total energy density of cavities with low γ shows important macroscopic quantum effects over quite large domains of volumes and temperatures.

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1. Introduction

The ideal classical cavity may be defined as a closed surface with a perfectly smooth and unitary reflection interior wall, where the discrete absorption and emission of quanta by the atoms are leading to the thermal equilibrium [1-3]. The quantum counterpart of this classical definition is the concept of an infinite potential well, ensuring a vanishing probability for the photon presence outside its surface. The cavity may be described by a Dirichlet boundary condition [4-6].

If we refer definitely to the black-body radiation, the effect of the geometrical confinement upon the frequency spectrum of the radiation stored inside the cavity may be assigned to an additional, kinetic quantizing, beyond that considered by Planck (referred to the discrete absorption and emission of quanta by the atoms of the cavity in view of reaching the thermal equilibrium). In this case, not only the energy exchanged between atoms and radiation is quantized (Planck's quanta), but also the radiation energy, through the agency of the discrete spatial directions of the allowed wave-vectors (as a result of the radiation confinement). This quantum device was named *double quantized cavity* (DQC)[7-11]. The effect of the additional energy quantizing is controlled by the adiabatic invariant $\gamma = TV^{1/3}$. DQC can be defined by a **low adiabatic invariant**, $\gamma \leq 1$ (this implying for instance small temperatures, in the proximity of about 1° K, and small volumes, in the proximity of about 1 cm^3), when the Planck spectrum of the black-body radiation presents a discrete pattern (of lines with irregular intensities), strongly depending on the cavity geometry [7-11].

The total energy density of cavities with low γ does not obey any longer the Stefan-Boltzmann law, but a new law, which includes a corrective factor depending on γ and imposes an exponential-type decrease to zero, for $\gamma \to 0$. The study of the geometrical confinement effect for cubic and spherical DQCs shows that, in spite of some additional complexity in the state identification, the total energy of the double quantized spherical cavity (DQSC) dependence [9-11] on the adiabatic invariant is almost identical to that of the double-quantized cubic cavity (DQCC).

2. Energy density of cubic cavity with low adiabatic invariant

The total energy density is written in classical (large adiabatic invariant) case as:

$$E = \sigma V T^4 \,, \tag{1'}$$

where $\sigma = 8\pi^5 k^4 / 15h^3c^3 = 7.56477 \cdot 10^{-15} \left[erg / cm^3 K^4\right]$ is Planck's constant of blackbody radiation energy (known by experimentalists as Stefan-Boltzmannn (S-B) constant). A possible universal form of this law, expressed in function of the normalized adiabatic invariant, is:

$$\frac{E}{kT} = \frac{8\pi^5}{15}\left(\frac{\gamma}{\alpha}\right)^3, \tag{1}$$

where $\gamma = TV^{1/3}$ is the adiabatic invariant and $\alpha = hc/k$ =1.4388 cm.K.

In DQCC, the total energy should be written by summing up the state energies up to the highest significant one, indexed by $q = q_T$ [10]:

$$E = \sum_{q=1}^{q_T} \frac{2g(q)\cdot h\upsilon_q}{\exp(h\upsilon_q/kT)-1} = \left[\frac{2k\alpha}{\gamma^4}\sum_{q=1}^{q_T}\frac{\sqrt{q}\cdot g(q)}{\exp(\alpha/\gamma)(\sqrt{q})-1}\right]\cdot VT^4$$
$$= \sigma_1 VT^4 \qquad (2')$$

or, in the universal form:

$$\frac{E}{kT} = \left[2\left(\frac{\gamma}{\alpha}\right)^{-4}\sum_{q=1}^{q_T}\frac{\sqrt{q}\cdot g(q)}{\exp(\alpha/\gamma)(\sqrt{q})-1}\right]\cdot\left(\frac{\gamma}{\alpha}\right)^3 \qquad (2)$$

We can write the total energy in DQCC as in S-B law, up to the corrective factor**:**

$$F\left(\frac{\gamma}{\alpha}\right) = \frac{\sigma_1}{\sigma} = \frac{15}{4\pi^5}\left(\frac{\gamma}{\alpha}\right)^{-4}\sum_{q=1}^{q_T}\frac{\sqrt{q}g(q)}{e^{(\alpha/\gamma)\sqrt{q}}-1}$$
$$= \frac{5.2515\cdot 10^{-2}}{\gamma^4}\sum_{q=1}^{109\gamma^2}\frac{\sqrt{q}g(q)}{e^{(1.4388/\gamma)\sqrt{q}}-1} \le 1 \qquad (3)$$

In the asymptotic limit, *g(q)* tends to $g_{asy}(q) = 2\pi\sqrt{q}$ (by averaging over many and very close modes), $F(\alpha/\gamma)$ tends to 1, and one arrives to the classical S-B law. The corrective factor is represented in Fig.1, in function of the adiabatic invariant.

The corrected S-B "constant" is down limited by the lowest cavity mode to the value: $\sigma_1 = 0.00178\,\sigma = 1.346\cdot 10^{-17}\left[erg\cdot cm^{-3}\cdot K^{-4}\right]$ and arrives to the asymptotic (upper limit) at the conventionally selected value $LT \approx 1$, for which: $\sigma_1 = 0.9703\ \sigma = 7.340\cdot 10^{-15}\left[erg\cdot cm^{-3}\cdot K^{-4}\right]$.

Thus, *the total energy in DQCC has a stronger dependence on temperature than was predicted by Stefan-Boltzmann law* and more correctly, is dependent on the adiabatic invariant. As the cavity is emptied of states, its total energy is strongly decreasing according *a new law* derived in Eq.(2').

Calculating the ratio (σ_1 / σ) from Eq. (3) with the exact degeneracy [10] and with the asymptotic relation $g(q) \approx 2\pi\sqrt{q}$, we found out differences in the order of approximately 5·10-3, which are negligible in these calculations. Thus, we can assert that *the small number of states in cavity (up to q_T), at low $\gamma = LT$, plays the key role in calculating E and not the exact degeneracy.*

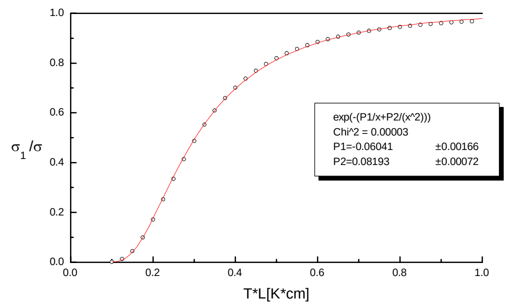
Therefore, ***the total energy density law of DQCC*** can be written in the form:

$$E \approx \left[\frac{15}{2\pi^4}\left(\frac{\alpha}{\gamma}\right)^4\sum_{q=1}^{q_T}\frac{q}{\exp(\alpha/\gamma)\sqrt{q}-1}\right]\cdot\sigma VT^4$$
$$= \left[0.32996\gamma^{-4}\sum_{q=1}^{109\gamma^2}\frac{q}{\exp(1.4388\,\gamma^{-1}\sqrt{q})-1}\right]\cdot\sigma VT^4 \qquad (4')$$

In universal form, Eq. (4') is:

$$\frac{E}{kT} \approx \left[4\pi\left(\frac{\gamma}{\alpha}\right)^{-4}\sum_{q=1}^{109\gamma^2}\frac{q}{\exp[\sqrt{q}/(\gamma/\alpha)]-1}\right]\cdot\left(\frac{\gamma}{\alpha}\right)^3 \qquad (4)$$

We have shown that the positions of the energy density peak and of HSL depend on the adiabatic invariant. Eq. (2') and the figure 1 show that the asymptotic limit can be set for $F(\alpha/\gamma) \approx 1 \rightarrow \gamma_{q\max} \approx 1[cm\cdot K] \rightarrow q_{\max} \approx 100$. On the other hand, the lowest cavity mode (1,0,0) imposes an inferior limit to the level number at $q_T = 1 \approx 109\gamma^2$ (the smallest frequency in the cavity) leading to $\gamma_{qmin} \approx$ 0.1 [cm.K].



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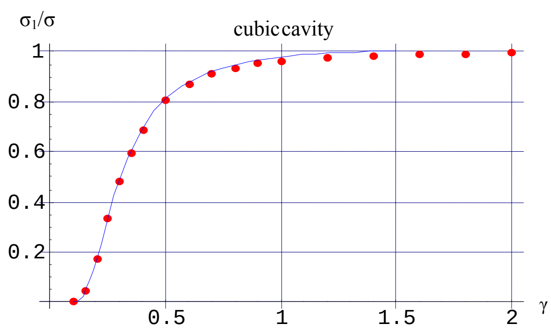

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Fig. 1. The corrective factor of the total energy in DQCC in function of the adiabatic constant γ = LT, calculated with: (a) exact weights (with dots) and fitted with the exponential from Eq. (6) (red curve); (b) analytical approximation of weights (with red dots) and fitted with the exponential from Eq. (6) (blue curve).

Thus, we can define the *double quantization regime* of the cubic cavity in the range:

$$1 \le q \le 100 \text{ or } 0.1 \le \gamma \le 1[\text{cm.K}] \quad (5)$$

We found out an exponential approximation for σ_1/σ as:

$$F(\gamma) = \sigma_1/\sigma \approx \exp[0.06/\gamma - 0.082/\gamma^2] \quad (6)$$

A reciprocity rule holds in DQCC: the cavity size and the temperature are reciprocal parameters, in the sense that the same effects (in the thermodynamics of the photon gas) can be obtained either by varying V or by varying T, if their product remains constant.

3. Energy density of spherical cavity with low adiabatic invariant

Using the kinetic energy states in a sphere with small radius R [9,11], we may derive the total energy of the black-body radiation inside the spherical cavity [10] as:

$$E = 2 \cdot kT\left(\frac{\alpha_1}{\gamma_1}\right) \sum_{q=1}^{q=\infty} \frac{z_q g_q}{\exp(\alpha_1/\gamma_1) z_q - 1}, \quad (7)$$

with $\gamma_1 = RT = 0.6203505\ \gamma$, $\alpha_1 = hc/2\pi k = \alpha/2\pi = 0.22899[cm \cdot K]$, $\alpha_1/\gamma_1 = 0.256556(\alpha/\gamma)$ and $z_q = 2\pi\nu_{nl}R/c$ (ν_{nl} are the zeros of the half-integer Bessel functions, characterizing the states of the spherical cavity) and g_q – the corresponding weights.

Similarly to DQCC, we can calculate the total energy in DQSC as in the classical law, up to a corrective factor, which takes in this case the form:

$$F(\gamma) = \frac{\sigma_1}{\sigma} = \frac{15}{4\pi}\left(\frac{\alpha_1}{\gamma_1}\right)^4 \sum_{q=1}^{q_T} \frac{z_q g_q}{\exp(\alpha_1 z_q/\gamma_1) - 1} = \frac{1.7405697 \cdot 10^{-2}}{\gamma^4} \sum_{q=1}^{q_T} \frac{z_q g_q}{\exp(0.57982897 z_q/\gamma) - 1} \quad (8)$$

It may be proved that:

$$\lim_{\gamma \to \infty} F(\gamma) = 1. \quad (9)$$

The demonstration [9,11] is based on the asymptotic degeneracy of the N – multiplets. The convergence of the function F toward 1 also resorts to the fact that the ratio of the dark state number to the allowed state number approaches zero for $N \to \infty$. The existence of allowed states beyond the (parabolic) barrier of the highest poles and the zigzag diagonal line is essential in reaching the aforementioned convergence.

The summation in Eq.(8) can be taken up to the index, q_T, corresponding to the maximum significant state (frequency) found in [9], $z_M = 48\ RT$. Thus, for example, in the correction calculation at $RT = 0.5$, the upper limit in the sum can be taken $q_T = 70$.

In Fig. 2, we have plotted the corrective function *F(γ)* (with solid line) in the interval $0 \le \gamma \le 2$, in which the truncation errors remain acceptable, for the limited number of calculated and ordered states (300). One can remark that this "exact" corrective factor imposes a faster decrease of the cavity energy than that predicted by the Stefan-Boltzmann law, as $\gamma \to 0$.

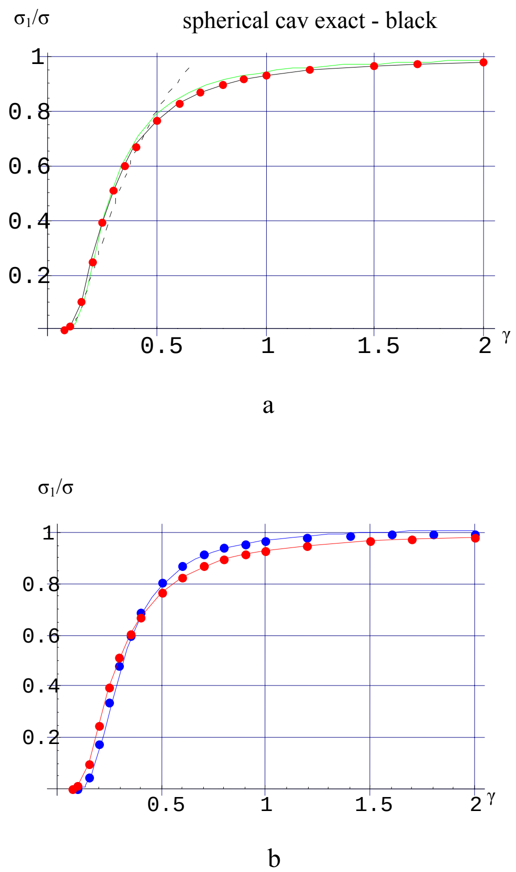


Fig. 2. (a) The corrective factor of the total energy density in DQSC in function of the adiabatic constant γ = LT: the exact calculation with dashes; the calculation with Eq.(11) with red dots joined by lines; the calculation with approximation from Eq.(12). (b) Comparison of the corrective factors for DQCC (blue) and DQSC (red).

We can write *the total energy density law of DQSC* in the form:

$$E = \left[\frac{15}{4\pi}\left(\frac{\alpha_1}{\gamma_1}\right)^4 \sum_{q=1}^{q_T} \frac{z_q g_q}{\exp(\alpha_1/\gamma_1) z_q - 1}\right] \cdot \sigma V T^4 = \sigma_1 V T^4 = \left[\frac{1.7405697 \cdot 10^{-2}}{\gamma^4} \sum_{q=1}^{q_T} \frac{z_q g_q}{\exp(0.57982897 z_q/\gamma) - 1}\right] \cdot \sigma V T^4. \quad (10)$$

Calculating the ratio (σ_I / σ) from Eq. (8) with the exact degeneracy [9,11] and with the analytic relation *g(q)* $\approx q^2$-1, we found out negligible differences in these calculations. Therefore, we can write the ***total energy density law of DQSC*** in the final form:

$$E \approx \left[\frac{15}{4\pi} \left(\frac{\alpha_1}{\gamma_1} \right)^4 \sum_{q=1}^{q_T} \frac{q(q^2-1)}{\exp(\alpha_1 q/\gamma_1)-1} \right] \cdot \sigma V T^4 =$$

$$= \left[\frac{1.31670094 \cdot 10^{-16}}{\gamma^4} \sum_{q=1}^{q_T} \frac{q(q^2-1)}{\exp(0.57982897 q/\gamma)-1} \right] \cdot VT^4$$

$$= \sigma_1 V T^4 \qquad (11')$$

The universal form of this law is:

$$\frac{E}{kT} \approx \left[8\pi \cdot 4.33243 \cdot 10^{-3} \left(\frac{\gamma}{\alpha} \right)^{-4} \sum_{q=1}^{q_T} \frac{q(q^2-1)}{\exp\left[0.256556 \cdot (\gamma/\alpha)^{-1} \cdot q\right]-1} \right] \cdot \left(\frac{\gamma}{\alpha} \right)^3 =$$

$$= \left[0.108886 \cdot \sum_{q=1}^{q_T} \frac{q(q^2-1)}{\exp\left[0.256556 \cdot (\gamma/\alpha)^{-1} \cdot q\right]-1} \right] \cdot \left(\frac{\gamma}{\alpha} \right)^{-1}. \qquad (11)$$

In the case of DQSC, we found out also a convenient approximation for $F(\gamma)$ as:

$$F(\alpha/\gamma) = \sigma_1/\sigma \approx \exp[0.001/\gamma - 0.06/\gamma^2] \qquad (12)$$

The convergence in summing up in Eq.(8) can be increased by iterative subtractions of the dominant asymptotic terms, which leads us to a precise formula over a very broad domain of adiabatic constant:

$$F(\gamma) = \frac{90}{\pi^4} \cdot \frac{(0.579828)^4}{\gamma^4} \sum_{q=1}^{q_T} \left[\frac{1}{1-\exp(-0.579828 q/\gamma)} \right]^4 \cdot$$

$$\cdot \exp(-2 \times 0.579828 q/\gamma) = \frac{0.104433}{\gamma^4} \sum_{q=1}^{q_T} \left[\frac{1}{1-\exp(-0.579828 q/\gamma)} \right]^4 \cdot$$

$$\cdot \exp(-1.159656 q/\gamma) \qquad (13)$$

We can define the *double quantization regime* of the spherical cavity approximately in the same range of the adiabatic invariant as in the case of cubic cavity. The same *reciprocity rule* (as for DQCC) holds: the cavity size and the temperature are reciprocal parameters in the DQSC, i.e. the same effects (in the thermodynamics of the photon gas) can be obtained either by varying *R* or by varying *T*, if their product remains constant.

The dependence of the total energy density in DQCs, cubic and spherical, on the adiabatic invariant, is quite similar. Following some preliminary calculations, we can infer that, for cavities with almost equal dimensions in all directions, this law holds irrespective of their shape.

4. Measurement of total energy density of cavities with low adiabatic invariant

Designing good cavities with quantum features shows that spherical cavities could be achieved practically with optimum shape and parameters. Measuring the total energy of cavities with small adiabatic invariant, one have to calculate the maximum of Planck distribution, which was found in [9-11] to be approximately at the same frequency as for the classical ones and given by Wien displacement law:

$$f_M = 2.8214394\,(\mathrm{k/h})\mathrm{T} = 5.87896 \cdot 10^{10}\ \mathrm{T} \qquad (14)$$

Using again the highest significant level (HSL) in the cavity, $q_T \approx 109\{\gamma[\mathrm{cm.K}]\}^2$ [10], we can write the maximum of Planck distribution in terms of cavity state numbers, as:

$$q_M \approx 3.8454 \cdot \{\gamma[\mathrm{cm.K}]\}^2, \qquad (15)$$

and the the ratio between the HSL and the maximum level numbers, frequencies respectively:

$$q_T / q_M \approx 28 \text{ and } f_T / f_M = \sqrt{q_T / q_M} \approx 5.3. \qquad (16)$$

Thus, one can assert that *the significant bandwidth of the black body radiation is:* $B \approx 5.3 f_M$.

It is interesting to find the condition for the localization of an antiresonance (dark state) in the maximum of the Planck distribution of DQCC. Assuming that q_M = 7, one can derive $q_T \approx$ 200 (at the limit of the quantum regime); thus, the first antiresonant singlet will produce a “hole” in maximum of the energy density of DQCC radiation at:

$$LT \approx 1.35\left[cm \cdot K\right]. \qquad (17)$$

For *L* =1cm, one obtains: $T \approx$ 1.35K and $f_M \approx$ 79.3 GHz (corresponding to λ = 3.78 mm, which is not in the maximum of the energy density distribution, when represented in λ scale). The quality factor of the cavity at this frequency, $Q = f/\Delta f \approx 8\pi L^3/\lambda^3 \approx 465$, leads to a good discrimination of the first antiresonant mode.

We have defined the quantum regime of the studied cavities in (5) as:

$$1 < q < 109 \qquad (18)$$

$$\text{or} \quad 0.1 < \gamma < 1 \quad [\mathrm{cm.K}] \qquad (19)$$

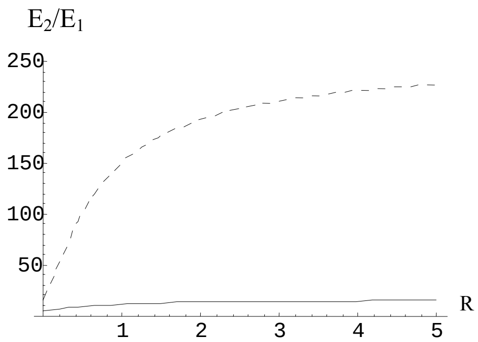
$$\text{or} \quad c^2/L^2 \le f^2 \le 109 c^2/L^2 \qquad (20)$$

Examples. One can see that DQCC with $\boldsymbol{L}$ **= 1cm** emits radiation in the frequency band $f \in [30, 300]$ GHz (at $T \in [0.1, 1]$K).

For $\boldsymbol{L}$ **= 1mm**, the quantum regime of the cavity asks for temperatures around that of the liquid He, $1 < T < 10$ and for $\boldsymbol{L}$ **= 20 μm**, the quantum regime of QCC occurs at temperatures around the room temperature: $50 < T < 500$ ($f_M \approx 17.7$ THz; $\lambda_M \approx 0.6c/f_M \approx 10\mu$m).

If we take $\gamma = 0.5$, the graphs from Fig. 2b show that $\sigma_1 / \sigma \approx 0.8$ and the total energy density of the cavity is $E = \sigma_1 VT^4 \approx 0.8\sigma VT^4 \approx 6 \cdot 10^{-15} VT^4 [erg]$. Thus, for a cavity with V = 1cm^3, the chosen adiabatic invariant imposes T = 0.5 K and one can derive $E \approx 6 \cdot 10^{-15} VT^4 [erg] = 0.375 \cdot 10^{-15} [erg] = 2.34 \cdot 10^{-4} eV$. This energy level is measurable with present instrumentation as in atomic spectroscopy (it is about hundred times smaller than Rydberg transition energies) and with cooled bolometers as the cosmic microwave background radiation (at the temperature of 2.7 K, with the peak at f_M ~ 160.4 GHz), where sensitivities of $\Delta T / T \le 2 \cdot 10^{-6}$ are accessible [12].

One can imagine two procedures for the measurement of the total energy density of spherical cavities with small adiabatic invariant based on energy ratios (in order to eliminate the systematic errors), as in Fig. 3.



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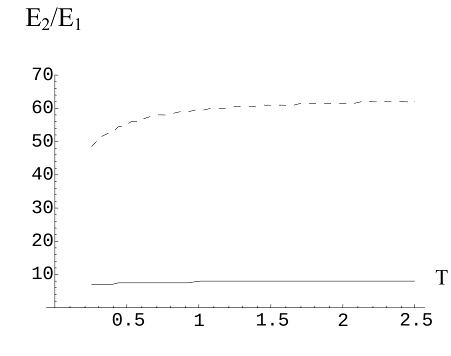

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Fig. 3. (a) The total energy ratio of two spherical cavities with equal radii, R, held at different temperatures (ratios), T_2 / T_1. (b) The ratio of the total energies of two spheres with two ratios of the chosen radii, held at the same temperature, T.

Considering the measurement of the total energy of a spherical cavity, with a diameter of 1cm, held at two temperatures, T_2 = 4K and T_1 = 1K, the ratio of its respective total energies (measured with highly sensitive bolometers) is 151.046. The graphs of the total energy ratio are plotted in Fig. 3a, for two different ratios T_2 / T_1 and for different values of the sphere radius, R; one can observe that the total energy ratio saturates in both cases, at very macroscopic radii.

Alternatively, one can measure the total energy ratio of two spheres with different radii, for example, R_2 = 1cm and R_1 = 0.5cm, held at the liquid He temperature, T = 4 K. The ratio of the corresponding total energies is 7.26217. In Fig. 3b, it is shown the dependence of this ratio on the temperature, for two ratios of the chosen radii; one can observe that the total energy ratio saturates in both cases, at low temperatures. These measurements can show important quantum effects at macroscopic scale.

5. Conclusions

The total energy of cavities with low values of the adiabatic invariants (obtained by summing up the exact contributions of the eigenvalues and their weights) does not obey any longer the Stefan-Boltzmann law. The new law, which is similar for cubic and spherical cavities, includes an exponential corrective factor depending on γ and imposes a faster decrease of the total energy to zero, for $\gamma \to 0$.

We have defined the double quantized regime both for cubic and spherical cavities by the conditions on the adiabatic invariants (or on principal quantum numbers): $0.1 \le \gamma \le 1$[cm.K]. The cavity volume and the temperature are reciprocal parameters in the sense that the same effects (in the photon gas) can be obtained either by varying V or by varying T, if their product remains constant. The limits of the double quantized regime are quite similar for cube and sphere, in terms of adiabatic invariant.

The total energy of the double quantized cavities shows large differences from the classical calculations over unexpected large intervals, which can be measured and show important macroscopic quantum effects.

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